Calculus
Chapter 11, etc.
2002 – 2003

Dy 1-4 exponential gwth
Dy 5 Newton's method
Dy 6,7,8 polar graphing
Dy 9 Analytic Geometry
Dy 10 Straight Lines
Dy 11 Conic Sections
Dy 12 Santa

Exam Ch 6
Applications
12/ 12

1 Pg 441
1 - 15
Implicit Diff
12/ 13

2 Pg 442
20, 26 – 28 Skip 26 "c"

More Implicit
12/16

3 Pg 448-449 1 – 7
Do problem 3C two ways.
1) use $\tan\theta = \frac{x}{3}$
2) use $\theta = \tan^{-1}\frac{x}{3}$
Related Rates 12/17

4 Pg 402 1 – 5, 9 – 11, 13

Parametric Equations
12/18

5 Pg 468
GQ Ch 11 1 – 4
Pg 469 RE 5 – 7
on pblm 5 find y' at
$(\pi, 1)$ only. 12/19
Review of rel rates & para

Minimum Day
12/ 20

During Christmas Vacation you are to prepare the Twelve Days of Christmas (on the back of this HW sked. Ho Ho Ho.)
 # 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,17

Vacation HW is due Jan 6. If you are unable to be in class then your work must be postmarked on or before Jan 6th.

Turn in
12 Days of Christmas
"Electric Eccentricity"
18, 19, 20, 21
1/ 6

Continue on Electric Eccentricity

MacWorld Expo 1/7

MacWorld Expo 1/ 8

turn in
"Electric Eccentricity" tomorrow
1/ 9

22 Pg 469 9–14
on pblm 11 find
y' at (1, 2) and y' at (1, 1)
Review Ex 1/10

23 Pg 469 – 470
15,16, 18, 19, 22,
24 –27, 29
Review Ex
1/13

24 Pg 396
2 – 16 even 20 – 30 even

More Polor Graphing
1/14

25 Pg 923 1 – 7

eccentrincity.
Conics in Polar notation
1/15

Put last year's exam on the boards.
Come to class
PREPARED!!
1/16

Chapter 11 etc. EXAM

School Holiday
1/ 20

Exams Return. Quarter grades issued.
1/21

Final Exam
Per 1, 2
1/22

Final Exam
Per 3, 4
1/23

Final Exam
Per 5, 6
1/24

The Twelve Days of Christmas

Each of the following homework assignments are to be prepared during the Christmas vacation. You will have to write dialogs of class lectures, including example problems for two topics. As most students have said on more than one occasion, "The book just doesn't present the material in an understandable manner." Here is your chance to show that *you can do* a better job. *Do honors level work!*

Day One ***Day Two***

Write a complete lecture concerning everything there is to know about the topics on pages 183 - 189. You are allowed to use the same examples as those which are in the book however, you are expected to clearly explain their significance in your own words.

Day Three

Demonstrate your understanding of the topics from Days One and Two by doing the following problems: Page 189 1 - 6, 8, 11, 18, 19.

Day Four

Write "narratives" covering every step of each problem in the problem set from Day Three.

Day Five

Do problems: Page 458 1, 2, 5, 7, 10

Day Six ***Day Seven***

Write a lecture covering the topics on pages 392 - 395. Tell everything there is to know about the topics! Use good examples.

Day Eight

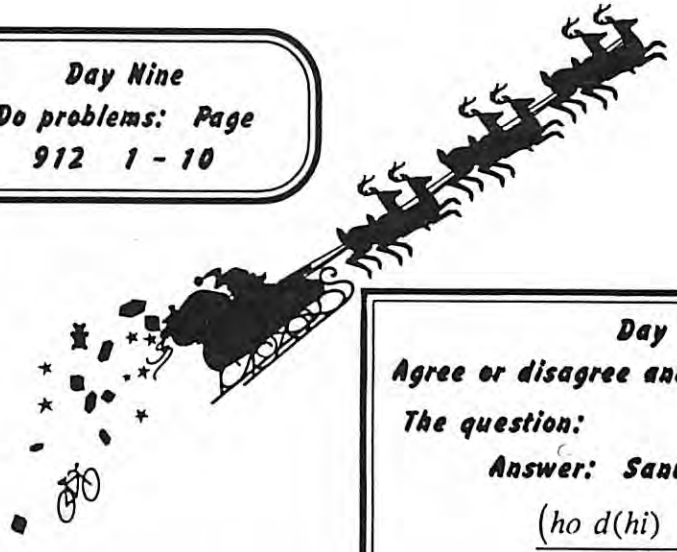
Do Page 396 1 - 17 odd, 21 - 31 odd

Day Nine

Do problems: Page 912 1 - 10

Day Ten

Do Page 914 - 915 1 - 10



Day Eleven

Do problems: Page 920 - 921 1 - ~~12~~

Day Twelve

Agree or disagree and justify, of course:

The question: What is $d(hi \,/\, ho)$***?***

Answer: Santa Says:

$$\frac{(ho\ d(hi) - hi\ d(ho))}{(ho\ ho)}$$

Implicit differentiation

Lecture
P. 437-441

Implicit differentiation is used whenever it may be difficult or impossible to only have one variable in a fctn which we wish to maximize (for example)

Example 1

Consider $x^2+y^2=25$ find $\frac{dy}{dx}$ at (3,4)

Solve for y

$y = \pm\sqrt{25-x^2}$ Notice the $\pm$, we must consider the + because (3,4) is in the 1ST Quadrant,

Now we have $y = \sqrt{25-x^2}$

using the chain rule $\frac{dy}{dx} = \frac{-2x}{2\sqrt{25-x^2}} = \frac{-x}{\sqrt{25-x^2}}$

So $\left.\frac{dy}{dx}\right\}_{x=3} = \frac{-3}{\sqrt{25-9}} = -\frac{3}{4}$.

Compare to
$x^2+y^2=25$
$2x+2y\frac{dy}{dx}=0$
$\frac{dy}{dx} = -\frac{x}{y}\Big| = -\frac{3}{4}$
$y=4$ and

Example 2

Given $z = \sqrt{x^2+a^2}$ where a is a constant, find $\frac{dy}{dx}$. we don't even have a y!!

where $\frac{dy}{dz} = \sqrt{x^2+a}$

what ever y is we know $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$

we can find $\frac{dz}{dx} = \frac{x}{\sqrt{x^2+a^2}}$ $\therefore \frac{dy}{dx} = \frac{x}{\sqrt{x^2+a^2}}\cdot\frac{dy}{dz}$

we still do not know what y is but since we know $\frac{dy}{dz}$ we find $\frac{dy}{dx}$ to be x.

"We have been using this all along: Chain Rule: derivatives inside."

in these situations:

$$x^2 + xy = 3 \qquad y + x\frac{dy}{dx} = 0$$

$$2x + x\frac{dy}{dx} + y$$

Lecture for implicit diff

we have always used chain rule —

$$y = 3x$$

$$\frac{dy}{dx} = 3\frac{dx}{dx}$$

$$\frac{dy}{dx} = 3$$

$$x^2 + xy = 3$$

$$2x\frac{dx}{dx} + y\frac{dx}{dx} + x\frac{dy}{dx} = 0$$

↑

most common omission

Lecture
Pg 437-441

~~example 3~~ already shown with

Going back to our original problem, $x^2+y^2=25$

and solving for $\frac{dy}{dx}$ implicitly we have (using chain rule)

$$2x\frac{dx}{dx}+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{x}{y}\Bigg\}_{\substack{x=3\\y=4}} = -\frac{3}{4}.$$

Do only this example

Lets consider a little more complicated pblm

example 4

$x^2+xy+y^5=3$. find equations of tangent at $(1,1)$

$$2x+y+x\frac{dy}{dx}+5y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}(x+5y)=-(y+2x)$$

$$\frac{dy}{dx}=\frac{-(y+2x)}{x+5y}\Bigg\}_{(1,1)} = -\frac{3}{6}=-\frac{1}{2}$$

So required equation has $(1,1)$ & $m=-\frac{1}{2}$

So $y-1=-\frac{1}{2}(x-1) \Rightarrow y=\frac{1}{2}x+\frac{3}{2}$

example 5

Given $y=\dfrac{\sqrt[3]{x}\sqrt{(1+x^2)^3}}{x^{4/5}}$ clearly this is a mess.

Consider

$$\ln y=\frac{1}{3}\ln x+\frac{3}{2}\ln(1+x^2)-\frac{4}{5}\ln x$$

Now $\frac{d}{dx} \Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{1}{3x}+\frac{3\cdot 2x}{2(1+x^2)}-\frac{4}{5x}$

So $\frac{dy}{dx}=\dfrac{\sqrt[3]{x}\sqrt{(1+x^2)^3}}{x^{4/5}}\left[\frac{1}{3x}+\frac{x}{1+x^2}-\frac{4}{5x}\right]$

Pg 3

lecture

Pg 437 – 441

example 6

The tin can problem.

$V = \pi r^2 h = 100$

$S = \underbrace{2\pi r^2}_{\text{top \& bottom}} + \underbrace{2\pi r h}_{\text{surface (ie sides of can)}}$

$$\frac{dS}{dr} = 4\pi r + 2\pi r \frac{dh}{dr} + 2\pi h$$

Since $\pi r^2 h = 100$

$$\frac{dh}{dr} \Rightarrow 2\pi r h + \pi r^2 \frac{dh}{dr} = 0$$

$$\frac{dh}{dr} = -\frac{2h}{r}$$

$$\frac{dS}{dr} = 4\pi r + 2\pi r\left(-\frac{2h}{r}\right) + 2\pi h$$

$$= 4\pi r - 4\pi h + 2\pi h$$

$$= 4\pi r - 2\pi h = 2\pi(2r - h) \overset{\text{set}}{=} 0$$

$\therefore r = \frac{h}{2}$ clearly if $r < \frac{h}{2}$ $S' < 0$; $r > \frac{h}{2}$ $S' > 0$ ✓

Therefore we do indeed have min surface area, for this specific pblm

$2r = h$ $\quad \pi r^2 h = 100$

$2\pi r^3 = 100 \Rightarrow r^3 = \frac{50}{\pi} \Rightarrow r = \sqrt[3]{\frac{50}{\pi}}$

$r = \frac{h}{2}$ and $h \Rightarrow \pi r^2 h = 100 \quad \pi \frac{h^3}{4} = 100 \Rightarrow h = \sqrt[3]{\frac{400}{\pi}} =$

$2\sqrt[3]{\frac{50}{\pi}}$

Compare the amt of work required using this method to the methods used in Chapt 6.

Using implicit diff relieves use of having to work with only one independent variable.

① $xy = 4 \quad (1, 4)$ $\qquad y = \frac{4}{x}$

$x\frac{dy}{dx} + y = 0$ $\qquad y' = -4x^{-2}\Big|_{x=1} = -4$

$\frac{dy}{dx} = -\frac{y}{x}\Big|_{1,4} = -4$

③ See Next Page

$x^2y + xy^2 = 12 \quad (3, 1)$

$x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 = 0$

$\frac{dy}{dx} = \frac{-(2xy + y^2)}{x^2 + 2xy}\Big|_{(3,1)} \Rightarrow \frac{-(6+1)}{9+6} = -\frac{7}{15}$

$xy^2 + x^2y - 12 = 0$

$y = \frac{-x^2 \pm \sqrt{x^4 + 48x}}{2x}$ etc See Next Page

⑤ $\frac{2xy}{\pi} + \sin y = 2 \quad (1, \frac{\pi}{2})$

$\frac{2x}{\pi}\frac{dy}{dx} + \frac{2y}{\pi} + \cos y\frac{dy}{dx} = 0$

$\left(\frac{2x + \cos y}{\pi}\right)\frac{dy}{dx} = \frac{-2y}{\pi}$

$\frac{dy}{dx} = \frac{-2y}{2x + \cos y}\Big|_{(1,\frac{\pi}{2})} = \frac{-\pi}{2+0} = \frac{-\pi}{2}$

6 $xy + \ln y = 1 \quad (1,1)$

$$x\frac{dy}{dx} + y + \frac{1}{y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y}{x + \frac{1}{y}} = -\frac{1}{2}$$

7 $2y^3 + 4xy + x^2 = 7 \quad (1,1)$

$$6y^2\frac{dy}{dx} + 4x\frac{dy}{dx} + 4y + 2x = 0$$

$$\frac{dy}{dx} = \frac{-2(x+2y)}{6y^2+4x}\Bigg]_{1,1} = \frac{-2(3)}{6+4} = -\frac{3}{5}$$

8 $e^y = \sin(x+y) \quad \left(\frac{\pi}{2}, 0\right)$

$e^y = \sin x \cos y + \sin y \cos x$

$\ln e^y = \ln(\quad)$

$y = \ln(\quad)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos(x+y)\left[1+\frac{dy}{dx}\right]}{\sin(x+y)}$$

Too hard. Try again

$e^y = \sin(x+y) \quad \left(\frac{\pi}{2}, 0\right)$

$$e^y\frac{dy}{dx} = \left(1+\frac{dy}{dx}\right)(\cos(x+y)$$

$$\frac{dy}{dx}\left[e^y - \cos(x+y)\right] = \cos(x+y)$$

$$\frac{dy}{dx} = \frac{\cos(x+y)}{e^y - \cos(x+y)}\Bigg]_{\frac{\pi}{2},0} = \frac{0}{e^{\frac{\pi}{2}} - 0} = 0$$

Pg 441 #3

$x^2y + xy^2 = 12 \quad (3,1)$

$xy^2 + x^2y - 12 = 0$

Quadratic in y

Find $\frac{dy}{dx}$ two ways

$$y = \frac{-x^2 \pm \sqrt{x^4 + 48x}}{2x} \Bigg| = \frac{-9 \pm \sqrt{81+144}}{6}$$

$x=3$

$\frac{-9 \pm 15}{6} \quad \frac{-9+15}{6} = 1$

use Plus because when $x=3$ $y=1$

$$y' = \frac{1}{4x^2}\left[2x\left[-2x + \frac{4x^3+48}{2\sqrt{x^4+48x}}\right] - 2\left(-x^2 + \sqrt{x^4+48x}\right)\right]$$

at $x=3$

$$y' = \frac{1}{36}\left[6\left[-6 + \frac{4\cdot 27 + 48}{2\cdot 15}\right] - 2(-9+15)\right]$$

$$= \frac{1}{36}\left[6\left[-6 + \frac{2\cdot 27 + 24}{15}\right] - 2(6)\right]$$

$$\frac{6}{36}\left[-6 + \frac{54+24}{15} - 2\right]$$

$\frac{120}{78}$ $\frac{78}{42}$

$$\frac{1}{6}\left[-8 + \frac{78}{15}\right]$$

$$\frac{1}{6}\left[\frac{-120+78}{15}\right]$$

$$\frac{1}{6}\left[\frac{-42}{15}\right]$$

$$-\frac{7}{15}$$

Implicit:

$x^2y + xy^2 = 12$

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(2xy+y^2)}{x^2+2xy} \Bigg| = \frac{-(2\cdot 3 + 1)}{9+6} \quad (3,1)$$

$$= -\frac{7}{15}$$

(9) $y = (x^3-1)^{3/2}(x^5)$

$\ln y = \frac{3}{2}\ln(x^3-1) + 5\ln x$

$\frac{1}{y}\frac{dy}{dx} = \frac{3}{2}\cdot\frac{3x^2}{(x^3-1)} + \frac{5}{x}$

$\frac{dy}{dx} = \left[\frac{9x^2}{2(x^3-1)} + \frac{5}{x}\right]\left[(x^3-1)^{3/2}(x^5)\right]$

(11) $y = \left(\sqrt{\cos x}\sqrt[3]{\ln x}\right)^5$

$\ln y = 5\ln\left(\sqrt{\cos x}\sqrt[3]{\ln x}\right)$

$= 5\left[\ln\sqrt{\cos x} + \ln\sqrt[3]{\ln x}\right]$

$= 5\left[\frac{1}{2}\ln\cos x + \frac{1}{3}\ln\ln x\right]$

$\frac{1}{y}\frac{dy}{dx} = 5\left[\frac{-\sin x}{2\cos x} + \frac{1}{3x\ln x}\right]$

$\frac{dy}{dx} = \left[\frac{-5}{2}\tan x + \frac{5}{3x\ln x}\right]\left(\sqrt{\cos x}\sqrt[3]{\ln x}\right)^5$

(12) $y = x^{x^2}$

$\ln y = x^2\ln x$

$\frac{1}{y}\frac{dy}{dx} = 2x\ln x + x$

$\frac{dy}{dx} = x^{x^2}(2x\ln x + x)$

441

$e^u \Rightarrow e^u \frac{du}{dx}$

(15) $e^{xy} + 5\sin x + y = 1 \qquad f'(0) = ?$

$e^{xy}\left[y + x\frac{dy}{dx}\right] + \cos x + \frac{dy}{dx} = 0$

$ye^{xy} + (xe^{xy} + 1)\frac{dy}{dx} + \cos x = 0$

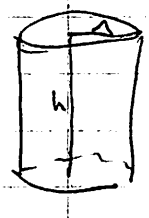
$\frac{dy}{dx} = \left.\frac{-(\cos x + ye^{xy})}{xe^{xy}+1}\right\}_{x=0} = \frac{-(1+y)}{1}$

$\frac{-(1+y)}{1} \Rightarrow \quad f(0) = 1 + 5\cdot 0 + y = 1$

$y = 0$

$\therefore -1$

(20)



$2\pi r + h = 100$

$2\pi + dh = 0$

$V = \pi r^2 h$

$\frac{dV}{dr} = h(2\pi r) + \pi r^2 \frac{dh}{dr}$

$\frac{h}{r} = \pi$

$= (100 - 2\pi r)(2\pi r) + \pi r^2(-2\pi)$

$= 200\pi r - 4\pi^2 r^2 - 2\pi^2 r^2$

$= 200\pi r - 6\pi^2 r^2$

$= 2\pi r(100 - 3\pi r) \overset{\text{set}}{=} 0$

$r = \frac{100}{3\pi}$ $\qquad \frac{\frac{100}{3}}{\frac{100}{3\pi}}$

$2\pi\left(\frac{100}{3\pi}\right) + h = 100$

$\frac{200}{3} + h \quad h = \frac{100}{3}$ $\qquad \pi$

26 $f(x) = y$ $\quad x^3y + y^2 = 2 \quad (1,1)$

a) WTS $x^3y' + 3x^2y + 2yy' = 0$

Take der of given equation. This is the result.

b) $3x^2y' + x^3y' + 3x^2y' + 6xy + 2yy'' + 2(y')^2 = 0$

c) $6x^2y'$

$$y' = \frac{-3x^2y}{x^3+2y}\Big]_{(1,1)} = \frac{-3}{3} = -1$$

$$6x^2\left[\frac{-3x^2y}{x^3+2y}\right] + (x^3+2y)y'' + 6xy + 2\left(\frac{-3x^2y}{x^3+2y}\right)^2 = 0$$

$$\frac{1}{x^3+2y}\left(\frac{-18x^4y}{x^3+2y} + 6xy + \frac{18x^4y^2}{(x^3+2y)^2}\right) = y''$$

$$\frac{1}{x^3+2y}\left(\frac{-18x^4y(x^3+2y) + 6xy(x^3+2y)^2 + 18x^4y^2}{(x^3+2y)^2}\right)$$

$$\frac{6xy(-3x^3(x^3+2y) + (x^3+2y)^2 + 3x^3y)}{(x^3+2y)^3}$$

$$\frac{6xy(-3x^6 - 6x^3y + x^6 + 6x^3y + 4y^2 + 3x^3y)}{(x^3+2y)^3}$$

$$6xy(-3x^6 + x^6 + 4y^2 + 3x^3y)$$

$$6(-1) + 3y'' + 6 + 2(1) = 0$$

$$3y'' = -2$$

$$y'' = -\frac{2}{3}$$

During discussion of #4.

1. Let it be shown —
   Assume they show use of chain rule

2. SAY "Since there has never been a typo in this book, we have no reason to doubt this method ...?"

3. Look $\dot{x}y = \frac{dy}{dt}$ so $\ddot{y} = \frac{d^2y}{dt^2}$ } $\frac{d^2y}{d^2x}$

$\dot{x} = \frac{dx}{dt}$ so $\ddot{x} = \frac{d^2x}{dt^2}$

is this 2nd der of y wrt x?

NO!!

Related Rates.

Students to prepare lectures on eqs 2-4.
Daniel Present Eg #1

Plan 3

Do Section B & C did 2 ways

① $\tan\theta = \frac{x}{3}$

↓

$\sec^2\theta \frac{d\theta}{dt}$ etc

② $\theta = \tan^{-1}\left(\frac{x}{3}\right)$

etc.

Pg 445-446

Do eg #3 as Demo

Method

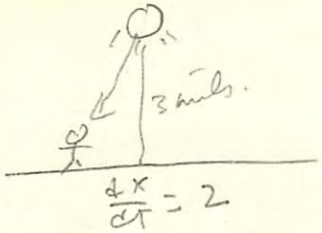
① draw picture if possible Label constants with numbers

② write Given: ---- Be careful NOT to label Evaluation pts.

③ write WTF: $\frac{dx}{dt}$; $\frac{d\theta}{dt}$

④ circle Top right Letter. Create Identity involving those letters.

⑤ Answer the question using a sentence.

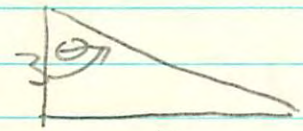


(3)

a) rate decreases as man gets further down the beach - it takes less turning to keep up with him.

b)

(1)



$$\tan\theta = \frac{x}{3}$$

$$\frac{d(\tan\theta)}{dt} = \frac{d\left(\frac{x}{3}\right)}{dt}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{3}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{3}\cdot 2 \cdot \cos^2\theta = \frac{2}{3}\cos^2\theta$$

This is unclear about how the rotation changes as $\dot{x} \to \infty$.

(2) $\theta = \tan^{-1}\frac{x}{3}$

$$\frac{d\theta}{dt} = \frac{1}{3}\frac{dx}{dt}\,\frac{1}{1+\frac{x^2}{9}}$$

$$\frac{2}{3}\cdot\frac{3}{\frac{9+x^2}{9}}$$

$$\frac{d\theta}{dt} = \frac{6}{9+x^2}$$

as x increases clearly, $\frac{d\theta}{dt}$ slows.

(5)

$\frac{dV}{dt} = 1000$ $\quad r = h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi h^3$

$\frac{dh}{dt} = ?$

$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$

$\frac{1000}{\pi h^2} = \frac{dh}{dt}$

a) $\frac{5}{\frac{1000}{\pi \cdot 2\pi}} = \frac{dh}{dt} = \frac{5}{2\pi} = .79577$

b) $\frac{10^3}{10^4\pi} = \frac{1}{10\pi}$

---

(6)

$\frac{dx}{dt} = 5$ $\quad \frac{dh}{dt} = ?$

$\frac{dy}{dt} = -6$

$h^2 = x^2 + y^2$

$2h\frac{dh}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

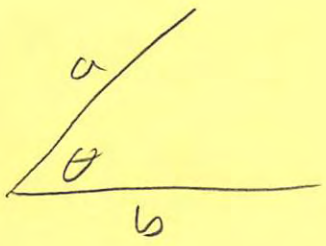
$\frac{dh}{dt} = \frac{5x - 6y}{\sqrt{x^2+y^2}}$

(3,4)

$= \frac{15 - 24}{5}$ hypot decreasing at

$\frac{dh}{dt} = -\frac{9}{5}$ $\frac{9}{5}$ sq ft/sec

(7)



$\frac{d\theta}{dt} = 1^\wedge/\text{sec}$

$\frac{da}{dt} = 3 \qquad \frac{db}{dt} = -2$

$\text{Area} = \left(\frac{1}{2}ab\right)(\sin\theta)$

$\frac{dA}{dt} = \sin\theta\left[\frac{1}{2}b\frac{da}{dt} + \frac{1}{2}a\frac{db}{dt}\right] + \frac{1}{2}ab\cos\theta\frac{d\theta}{dt}$

$= \frac{\sqrt{2}}{2}\left[\frac{1}{2}\cdot 5(3) + \frac{1}{2}(4)(-2)\right] + \frac{1}{2}(4\cdot 5)\frac{\sqrt{2}}{2}\cdot 1$

$= \frac{\sqrt{2}}{2}\left[\frac{15-8}{2}\right] + \frac{20\sqrt{2}}{4}$

$\frac{7\sqrt{2}}{4} + \frac{20\sqrt{2}}{4}$

$= \frac{27\sqrt{2}}{4}.$

8 $\dot{x} = 3 \quad \dot{y} = 4 \quad \ddot{x} = -2 \quad \ddot{y} = 5$

a) $\frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dx}$

$4 \cdot \frac{1}{3} = \frac{4}{3} = \frac{dy}{dx}$

b) $\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\left(\frac{\dot{y}}{\dot{x}}\right)}{dx}$

leads to

$\frac{d\left(\frac{\dot{x}}{\dot{x}}\right)}{dt} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$

but wrt x we get

$\frac{d\left(\frac{\dot{y}}{\dot{x}}\right)}{dx} = \frac{d\left(\frac{\dot{y}}{\dot{x}}\right)}{dt} \Big/ \frac{dx}{dt}$

This comes the cubic

$\frac{3 \cdot 5 - (4)(-2)}{3^3} = \frac{15 + 8}{27} = \frac{23}{27}$

11 $x = e^t + t \qquad y = \sin t + t^2$

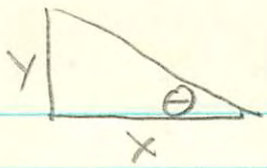
$\dot{x} = e^t + 1 \qquad \dot{y} = \cos t + 2t \qquad \frac{\dot{y}}{\dot{x}} \quad \frac{4}{r}$

$\ddot{x} = e^t \qquad \ddot{y} = -\sin t + 2$

$y' = \dfrac{\cos t + 2t}{e^t + 1} \qquad y'' = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3}$

$$y'' = \frac{(e^t+1)(-\sin t + 2) - e^t(\cos t + 2t)}{(e^t+1)^3}$$

15



a) $\tan\theta = \frac{y}{x}$

b) $x\tan\theta = y$

$\dot{\theta}\left(\dot{x}\sec^2\theta\right)$

$\tan\theta\,\dot{x} + x\sec^2\theta\,\dot{\theta} = \dot{y} \rightarrow \ddot{\theta}x\sec^2\theta + \dot{\theta}\left(\sec^2\theta\,\dot{x} + 2x\theta\dot{\theta}\sec^2\theta\tan\theta\right)$

c) $\ddot{\theta}\dot{x}\sec^2\theta + \tan\theta\,\ddot{x} + \quad = \ddot{y}$

$\ddot{x}\tan\theta + \sec^2\theta\left(2\dot{\theta}\dot{x} + \ddot{\theta}x + 2x\theta\dot{\theta}\tan\theta\right)$

16. Requires use of $\int$ to get Volume of kettle.

17. $\frac{dx}{dt} = 2\frac{dy}{dt}$ $\quad 2\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2}$

$\frac{dy}{dx} = \frac{1}{2}$

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}$

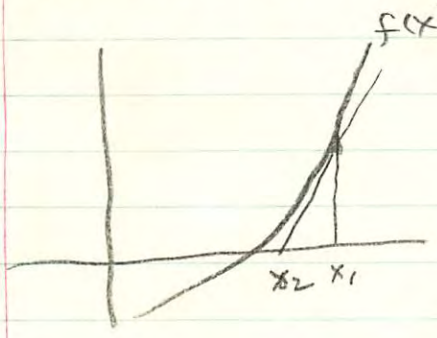
$\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3} \Rightarrow \frac{2\dot{x}\ddot{x} - \frac{1}{2}\dot{x}\ddot{x}}{(\dot{x})^3}$

$\frac{d^2y}{dx^2} = \frac{3}{2}\left(\frac{\ddot{x}}{(\dot{x})^2}\right)$ since there is NO information concerning $\ddot{x}$ and $\dot{x}$ this is as far as it will go.

Newton's method for solving an equation

455-458

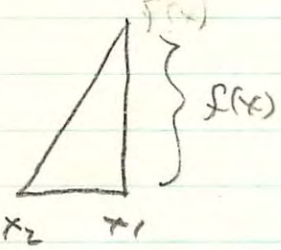
General situation



1ST we guess at solution say x_1.

from the picture we see x_1 is not the solution. the tangent to f at $(x_1, f(x_1))$ has an x intercept (provided $f'(x_1) \neq 0$).

this x intercept is a closer approximation to the root than is x_1.



$$\frac{f(x_1)}{x_1 - x_2} = \text{slope} = f'(x_1)$$

$$f(x_1) = (x_1 - x_2) f'(x_1) \Rightarrow$$

$$\frac{f(x_1)}{f'(x_1)} = x_1 - x_2$$

$$x_2 = x_1 - \frac{f(x_1)}{f(x_2)}$$

This is quicker than 1ST determining the equation of the tangent then determining the x intercept of that line.

ex 1. Estimate the square root of 3.

Let $x^2 = 3$ i.e. define $f(x) = x^2 - 3$ and solve.

So $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ w/ $x_1 = 1$

$= 1 - \frac{-2}{2}$

$x_2 = 2$

go again ↓

1	0	-3	
1	1	-2	f(1)
1	2		f'(1)

Pg 455-458

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

	1	0	−3
2	1	2	+1
2	1	4	

$$= 2 + \frac{+1}{4} = 1.75$$

etc.

$$= 1.75 - \frac{.0625}{3.5}$$

	1	0	−3
1.75	1	1.75	.0625
1.75	1	3.5	

$$= 1.75 - .01785\ldots$$

$$= 1.732\ldots$$

$\sqrt{3}$ actually $\Rightarrow$ 1.73205

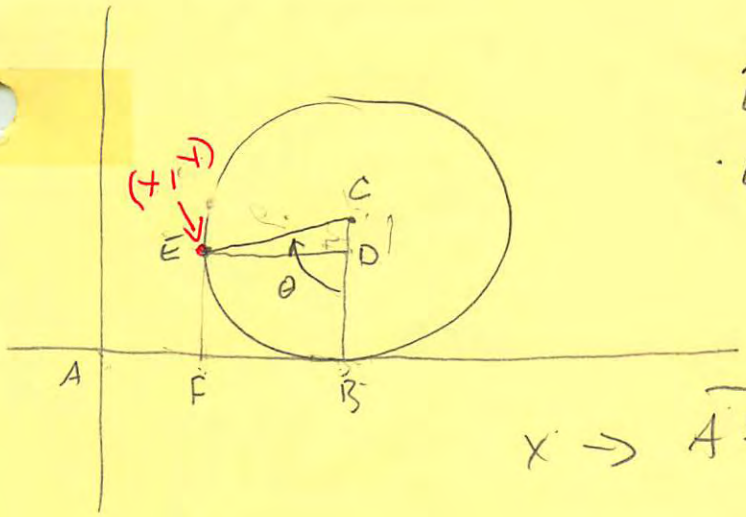
what if $f'(x) = 0 \quad \forall x \in D_f$? f is constant doesn't cross

what if $f'(x) \neq 0$, $f''(x) = 0 \ \forall x D_f$ f is linear and Newton's method will yield EXACT solution on 1st Approx.

if $f'(x) \neq 0$, $f''(x) \neq 0$ $f'(x)$ near zero. then f is nearly horizontal so tangent will not hit as close as desirable

for plsm 1 fctn Since $\sqrt{a}$ is the root of the $f(x) = x^2 - a$ Now put into Newton's Law,

Parametric equations. Pg 398



$\overset{\frown}{EB} = \overline{AB}$

$EC = \rho$ radius

$x \rightarrow \overline{Af} = \overline{AB} - \overline{ED}$

$\rho\theta - \rho\sin\theta$

same as $\overset{\frown}{EB}$

$y \rightarrow \overline{EF} = BC - CD$

$\rho - \rho\cos\theta$ $\quad \frac{adj}{Hyp} = \cos\theta$

So Position of track is

$$(x, y) = \begin{cases} x = \rho\theta - \rho\sin\theta \\ y = \rho - \rho\cos\theta \end{cases}$$

In TI 82

Mode PAR

Window $T_{min} = 0$ $\quad X_{min} = -1$ $\quad Y_{min} = -1$

$T_{max} = 20$ $\quad X_{max} = 20$ $\quad Y_{max} = 5$

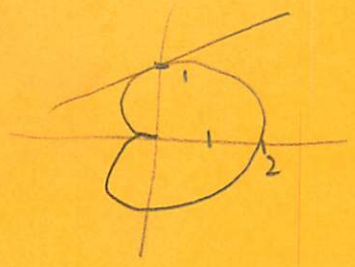
$T_{step} = .1$ $\quad X_{scl} = 3$ $\quad Y_{scl} = 1$

Put x into X_T let $\rho = 1$ and $\theta = T$ for calculations.

y into Y_T

#13

$r = 1 + \cos\theta$



a) $x = r\cos\theta \quad y = r\sin\theta$

$\therefore \left((\cos\theta + \cos^2\theta), (\sin\theta + \sin\theta\cos\theta)\right)$ (X, Y) in terms of θ

b) Slope of tangent line at (0, 1) (continued)

$\frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{dy}{dx}$

$\frac{dy}{d\theta} = \cos\theta + \cos^2\theta - \sin^2\theta$

$\frac{dx}{d\theta} = -\sin\theta - 2\cos\theta\sin\theta$

$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{0+0-1}{-1-0} = 1$

so Slope of curve is 1.

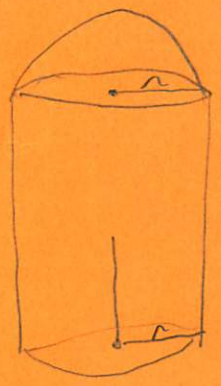
c) $\lim_{\theta \to 180^-} \frac{dy}{dx} = \lim_{\theta \to 180^-} \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos^2\theta\sin\theta}$ $\frac{0}{0}$ l'hospital

$\lim_{\theta \to 180^-} \frac{-\sin\theta - 2\cos\theta\sin\theta - 2\sin\theta\cos\theta}{-\cos\theta + 2\sin^2\theta - 2\cos^2\theta}$

$\frac{-0 - 0 - 0}{1 + 0 - 2} = \frac{0}{-1} = 0$

$\therefore$ as $\theta \to 180^-$ the curve goes into the origin horizontally.

3.



$V_{sphere} = \frac{4}{3}\pi r^3$

Surface bottom + sides + $\frac{1}{2}$ sphere

$S = \pi r^2 + 2\pi r h + 2\pi r^2$

$S = 3\pi r^2 + (2\pi r)h$

$0 = 6\pi r + h\,2\pi + 2\pi r \frac{dh}{dr}$

$V = \pi r^2 h + \frac{2}{3}\pi r^3$

$V = \pi r^2 (h + \frac{2}{3} r)$

so $\frac{dh}{dr} = \frac{-(6\pi r + 2\pi h)}{2\pi r}$

$= \frac{-3r - h}{r}$

$\frac{dV}{dr} = (h + \frac{2}{3}r)2\pi r + \pi r^2 \left(\frac{dh}{dr} + \frac{2}{3}\right)$

$= 2\pi r h + \frac{4\pi r^2}{3} + \pi r^2 \left(\frac{-3r - h}{r}\right) + \frac{2\pi r^2}{3} \overset{SV}{=} 0$

$2\pi r h + 2\pi r^2 - 3\pi r^2 - \pi r h = 0$

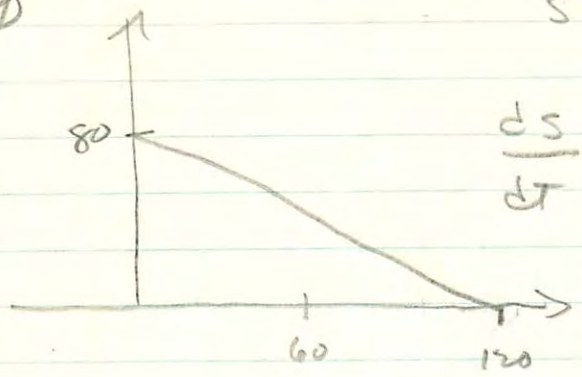
$-\pi r^2 + \pi r h$

$-\pi r (r - h) = 0$

$\therefore r = h.$

← Redo : min s^2 instead of s

(9)



$s = \sqrt{x^2+y^2}$ $\quad \frac{dx}{dt} = 60$ $\quad \frac{dy}{dt} = 80$

$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}} \Bigg] = \quad \begin{matrix} x=120 \\ y=80 \end{matrix}$$

$$\frac{120\cdot 60 + 80\cdot 80}{\sqrt{120^2+80^2}} =$$

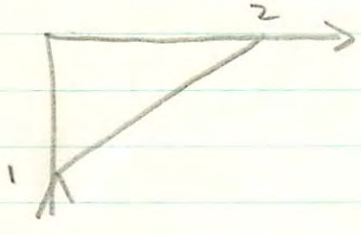
$\sqrt{(2^3\cdot 3\cdot 5)^2 + (2^4\cdot 5)^2}$ $\quad \frac{120\cdot 60 + 80\cdot 80}{} = \frac{40(3\cdot 60 + 2\cdot 80)}{}$

$2^3\cdot 5\sqrt{(3)^2+2^2} = 40\sqrt{13}$ $\quad = 40\sqrt{13}$

180
160
340

$\frac{340}{\sqrt{13}} = \frac{ds}{dt}$ dist between is increasing since $\frac{ds}{dt} > 0$

(10)



$\frac{dx}{dt} = 40$ $\quad \frac{dy}{dt} = -50$

(Neg because y distance is decreasing due to direction of travel.)

$s = \sqrt{x^2+y^2}$ ∴

$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}} \Bigg] = \quad \begin{matrix} x=2 \\ y=1 \end{matrix} \qquad \frac{2\cdot 40 - 50}{\sqrt{5}} = \frac{30}{\sqrt{5}} = 6\sqrt{5}$$

OR

$s^2 = x^2 + y^2$

$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$

$\sqrt{5}\frac{ds}{dt} = 2\cdot 40 + 1\cdot(-50)$ ∴ $\frac{ds}{dt} = \frac{30}{\sqrt{5}}$.

(11) $y^7 + xy = 2 \qquad x=1,\ y=1$

2nd der is $\frac{1}{3^4 4}$

$$7y^6\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

$$\left.\frac{dy}{dx} = \frac{-y}{7y^6+x}\right\}_{\substack{x=1\\y=1}} = -\frac{1}{8}$$

(12) $\ln(x+y) + x + x^2 + y = 2 \qquad x=1,\ y=0$

$$\frac{1+\frac{dy}{dx}}{x+y} + 1 + 2x + \frac{dy}{dx} = 0$$

$$1 + \frac{dy}{dx} + x + 2x^2 + y + 2xy + (x+y)\frac{dy}{dx} = 0$$

$$\left.\frac{dy}{dx} = \frac{-(1+x+2x^2+y+2xy)}{(1+x+y)}\right\}_{\substack{x=1\\y=0}}$$

$$= \frac{-(1+1+2)}{(1+1+0)} = -2 = \frac{dy}{dx}$$

(13) $y = (1+x^2)^3 e^{x^2} \sin^5 3x$

$$\ln y = 3\ln(1+x^2) + x^2 + 5\ln\sin 3x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{6x}{1+x^2} + 2x + \frac{15\cos 3x}{\sin 3x}$$

$$\frac{dy}{dx} = \left(\qquad\qquad\right) y$$

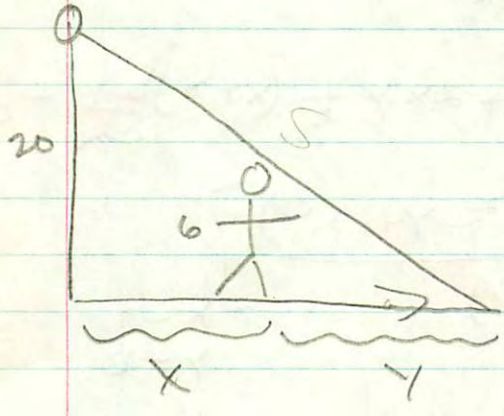
(14) $$y = \frac{(\sqrt{x^3+1})^5}{(x^2+1)^4}$$

$$\ln y = \frac{5}{2}\ln(x^3+1) - 4\ln(x^2+1)$$

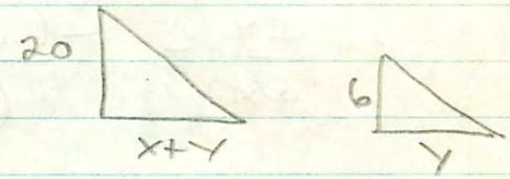
$$\frac{1}{y}\frac{dy}{dx} = \frac{15x^2}{2(x^3+1)} - \frac{8x}{x^2+1}$$

$$\frac{dy}{dx} = \left(\frac{15x^2}{2(x^3+1)} - \frac{8x}{x^2+1}\right)\frac{(x^3+1)^{5/2}}{(x^2+1)^4}$$

(15)



$$\frac{dx}{dt} = 5 \qquad \frac{dy}{dt} = ?$$



$$\frac{x+y}{20} = \frac{y}{6}$$

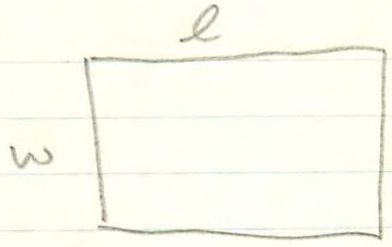
$$6x = 14y$$

$$6\frac{dx}{dt} = 14\frac{dy}{dt}$$

$$\frac{30}{14} = \frac{dy}{dt}$$

$$\frac{15}{7} = \frac{dy}{dt}$$

16



$\frac{dl}{dt} = 7$ $\quad l = 12$

$\frac{dw}{dt} = -3$ $\quad w = 5$

a) $A = wl$

$$\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt} \Rightarrow 12(-3) + 5(7) = -1$$

Area decreasing at rate of 1 sq ft/sec

b) $P = 2l + 2w$

$$\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt} \Rightarrow 2(7) + 2(-3) = 8$$

P is increasing at rate of 8 ft/sec.

c) $D = \sqrt{l^2 + w^2}$

$$\frac{dD}{dt} = \frac{l\frac{dl}{dt} + w\frac{dw}{dt}}{\sqrt{l^2+w^2}} \Rightarrow$$

$$\frac{12 \cdot 7 + 5(-3)}{\sqrt{12^2+5^2}} = \frac{84-15}{13} = \frac{69}{13}$$

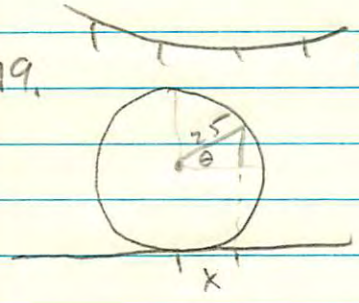
increasing at rate of $\frac{69}{13}$ ft/sec.

18. $S = 4\pi r^2 \quad V = \frac{4}{3}\pi r^3 \quad$ Given: $\frac{dV}{dt} = K 4\pi r^2$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2} = \frac{K 4\pi r^2}{4\pi r^2} = K$$

19.



$\frac{1}{10}$ rev/min $= \frac{1}{10} \cdot \frac{2\pi}{1 \text{ Rev}}$ /min $= \frac{\pi}{5}$ Rad/min

$\therefore \frac{d\theta}{dt} = \frac{\pi}{5}$ $\qquad \left.\frac{dx}{dt}\right| = ?$

$\theta = 30$ 2:00

$\theta = 60$ 1:00

$\cos\theta = \frac{x}{25}$

$$-\sin\theta \frac{d\theta}{dt} = \frac{1}{25}\frac{dx}{dt}$$

$$-25 \sin\theta \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$-5\pi \sin\theta = \left.\frac{dx}{dt}\right|_{\theta=30} = -\frac{5\pi}{2}$$

$$\left.\frac{dx}{dt}\right|_{x=60} = -\frac{5\pi\sqrt{3}}{2}$$ clearly this is faster

Finally consider $\frac{dx}{dt} = -5\sin\theta \frac{d\theta}{dt}$. Since the negative sign refers to direction we need only consider $\left|\frac{dx}{dt}\right|$ when we look at when the shadow is moving fastest & slowest. Since (Sin) is max at 90° & 270° and min at 0, 180 clearly shadow moves fastest at 3:00 position and slowest at 12:00 positions.

Show using Parameters on TI 82

Examples on Pg 397-398

Natural growth & decay. 6.4
183-191

The rate of growth or decay is proportional to the amount present at time t.

$$f'(t) = kf(t)$$

$f'(t)$: rate; k: constant proportion also known as "instantaneous growth or decay rate"; $f(t)$: in amount at time t.

growth or decay is exponential. So

$$f(t) = Ae^{kt}$$

eg 1 Pg 185 1977 $\Rightarrow$ 3.8 billion

$t=0$ A

growth rate is .02 $\Rightarrow k$ So.

$$f(t) = 3.8e^{.02t}$$

this can be simplified since $e^{.02}$ is constant.

$e^{.02} = 1.02$ So

$$f(t) = 3.8(1.02)^t$$

at $t=23$ (Year 2000, 23 yrs after 1977)

$$f(23) = 3.8(1.02)^{23} =$$

$$3.8(1.576899) \approx 6 \text{ Billion}$$

Notice $e^{.02} = 1.02 \Rightarrow 1 + \frac{2}{100} = 1 + 2\%$

This is a 2% increase per yr.

generally $e^k = 1 + \%$ increase.

eg 2 ↓

eg 2

1 pm → 100 gr

4:30 pm → 250 gr

? → 400 gr

Better method

$Y = Ae^{kT}$ → $400 = 100(1.3)^T$

$250 = 100(e^k)^{\frac{7}{2}}$

$4 = (1.3)^T$

$2.5 = (e^k)^{\frac{7}{2}}$

$\frac{\ln 4}{\ln 1.3} = T$

$(2.5)^{\frac{2}{7}} = e^k$

$T = 5.28 \approx 5$ hrs 20 min

$1.3 = e^k$

sample will have 400 gr at 6:20 pm

$\ln 1.3 = \ln e^k$

$.26 = k$

after return of 12 DAYS of Xmas ...
"Now people make exponential growth too complicated..."

Example 2 pg 184

$Y = Ae^{kt}$ $A = 100$ $t = 0$ (1pm)

$Y = 100e^{kt}$ at 4:30 $t = \frac{7}{2}$ $Y = 250$

$250 = 100e^{k(\frac{7}{2})}$

$\frac{250}{100} = e^{\frac{7}{2}k}$

$\left(\frac{5}{2}\right) = e^{\frac{7}{2}k}$

$\left(\frac{5}{2}\right)^{\frac{2}{7}} = e^{k}$ $\longrightarrow$

General equation

$Y = 100\left(\frac{5}{2}\right)^{\frac{2t}{7}}$

Notice that we did not solve for k.
we could have solved for k and would only when required to have "growth rate"

$Y = 100(1.299)^{t}$

eg 2 1pm ⇒ 100 grams
4:30pm ⇒ 250 grams. $3\frac{1}{2}$

$f(t) = Ae^{kt}$ $\frac{3}{7}\ln 2.5 = k$

$250 = f(\frac{7}{2}) = 100\, e^{k\frac{7}{2}}$

$2.5 = e^{k\frac{7}{2}}$

$(2.5)^{\frac{2}{7}} = e^{k}$ ~~$f(t) = 100(2.5)^{\frac{2t}{7}}$~~

$1.3 = e^{k}$ ~~solve for k~~ inc growth.

$f(t) = 100(2.5)^{\frac{2t}{7}}$

$400 = 100(2.5)^{\frac{2t}{7}}$

$4 = (2.5)^{\frac{2t}{7}}$

$4^{\frac{7}{2}} = (2.5)^{t}$

Too hard.

OR

$f(t) = 100e^{.26t}$

$400 = 100e^{.26t}$

$4 = 1.3^{t}$

$\frac{\ln 4}{\ln 1.3} = t \approx 5.28$

5 hrs 20 min

5 hrs 80 min.

$\frac{7}{2}\ln 4 = t \ln 2.5$

$5.30 = t$

eg 3 $t_{1/2} = 5730$ A = orig amt.

$f(t) = Ae^{kt}$

$\frac{A}{2} = Ae^{kt}$

$\frac{1}{2} = e^{kt}$.

$-\ln 2 = kt$

$\frac{-\ln 2}{5730} = k.$

$k \approx -.00012$

(3) Let today population $= z$

$e^k = 1.01$ 1% increase

$2z = z(1.01)^t$

$2 = (1.01)^t$

$\ln 2 = t \ln(1.01)$

$\frac{\ln 2}{\ln 1.01} = t$

① $f(\tau) = A e^{K\tau}$

$f(\tau) = 10 \cdot 3^{\tau}$ at $\tau = 0$ orig amt $\Rightarrow$ 10 grams.

$e^K = 3^{\tau}$

$K = \tau \ln 3$ at $\tau = 1$ $K = \ln 3$ growth rate.

~~e^K~~ $e^K = 1 +$ % increase ?

Pg 185 ex 1

$\frac{10 \cdot 3^{\tau+1}}{10 \cdot 3^{\tau}} = 3 \Rightarrow$ 1 + 2.00 ← % increase, 200% increase.
↑ orig amt

② $f(\tau) = 3 \cdot 2^{\tau}$

a) $12 = 3 \cdot 2^{\tau}$

$2^2 = 2^{\tau}$

$\tau = 2$

b) $5 = 3 \cdot 2^{\tau}$

$\frac{5}{3} = 2^{\tau}$

$\frac{\ln 5 - \ln 3}{\ln 2} = \tau = .737$

c) $\frac{3}{2} e^{K\tau} = \frac{3}{2} \cdot 2^{\tau}$

$K = \ln 2 \approx .693$

③ $f(\tau) = A e^{K\tau}$ $e^K = 1 +$ % increase.

$2Z = (Z)(1.01)^{\tau}$ let $Z =$ today's population

$2 = (1.01)^{\tau}$

$\frac{\ln 2}{\ln 1.01} = \tau \approx 69.7$ yrs. $\approx$ 69 yrs 8 months.

④

$11000 = 220 (1.01)^{\tau}$

$44.55 = (1.01)^{\tau}$

$\frac{\ln 44.55}{\ln 1.01} = \tau$

$\tau = 152$

1977 ← $\tau = 0$

152 ← τ

2129 yrs.

(5) $5^{0.3t} = 2$

$.3t = \frac{\ln 2}{\ln 5}$

$t = \frac{10}{3}\frac{\ln 2}{\ln 5}$

$t = 1.44$

(6) $f(t) = Ae^{kt}$

double 5 hr

$f(t) = 100e^{kt}$

8 hr triple

$f(10) = 100e^{10k}$

10 hr

$4 = e^{10k}$ $\quad 4 = 5^{10}$

$\ln 4 = 10k$

$.139 = k$

(8) $f(t) = Ae^{kt}$ $\quad f(0) = 4$ $\quad f(1) = 8$

a) $A = 4$ b) $k =$ c) $b = e^k$

$f(1) = 8 = 4e^k$

$2 = e^k$

$\ln 2 = k$

$b = 2$

(11) $f(t) = 10^t$

wts $f'(t) = c10^t$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$y = 10^t$

$\ln y = t \ln 10$

$\frac{dy}{dt}\frac{1}{y} = \ln 10$

$\frac{dy}{dt} = y \ln 10$

$f'(t) = 10^t \ln 10$

$\frac{dt}{dy} \cdot \frac{dy}{dt} = 1$

No because $f(t) = 10^t = e^{t \ln 10}$

where $k = \ln 10$, & $A = 1$

(15) combining the corollary pg 185 with the Thm pg 184.

a fctn which satisfies $f'(t) = kf(t)$ must be in the form of $f(t) = Ke^{kt}$.

we want $k=1$, when $K=1$ $e^k = e$!.

$f(t) = Ae^t$ (Thm pg 184).

(16)

a) $\frac{ds}{dt} = cs \Rightarrow f(s) = Ab^t = Ae^{kt}$.

(corollary pg 185)

b) $f'(s) = Ake^{kt}$ still exponential.

c) if initial speed is zero then $f(s)$ is a constant and then it does not increase exponentially.

(18) $\lim_{k\to 0} \frac{\ln(1+k)}{k} = \lim_{k\to 0} \frac{\ln 1+k - \ln 1}{k}$

$\lim_{k\to 0} \frac{\ln(x+k) - \ln x}{k} \Big\}_{x=1} \Rightarrow f(x) = \ln x$

$f'(x) = \frac{1}{x} \Big\}_{x=1} = 1.$

$\ln(1+k) = k$ for k small $\Rightarrow$

$\lim_{k\to 0} \frac{\ln(1+k)}{k} = 1$

19. $e^K \doteq 1+K \qquad K \to 0$

$$\lim_{K\to 0} \frac{e^K}{1+K} = 1$$

$\Downarrow$

goes directly to 1

or

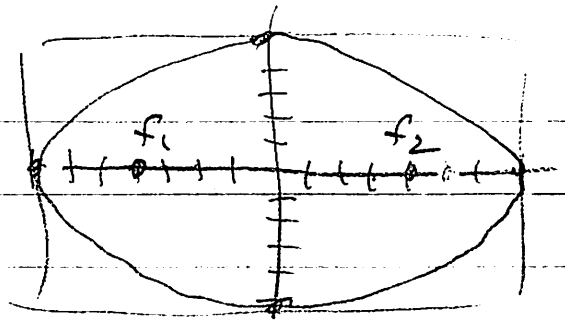
$$\lim_{K\to 0} \frac{e^K - 1}{K} = 1$$

$$\lim_{K\to 0} \frac{e^{0+K} - e^0}{K} = \lim_{K\to 0} \frac{e^{x+K} - e^x}{K} \quad \text{(with } x = 0\text{)}$$

$$f(x) = e^x$$

$$f'(x) = e^x \Big]_{x=0} = 1.$$

1. $\frac{x^2}{49} + \frac{y^2}{25} = 1$



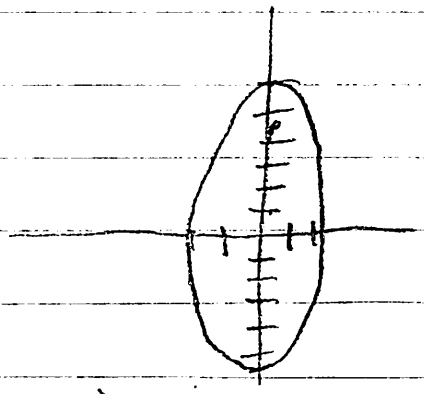
$c^2 = a^2 - b^2$

$c^2 = 49 - 25$

$= 16$

$c = \pm 4$

2. $\frac{x^2}{4} + \frac{y^2}{36}$

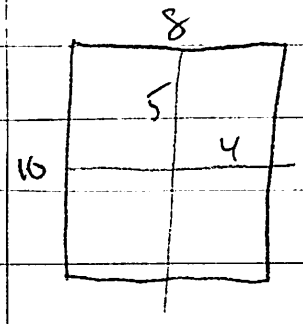


$c^2 = 36 - 4 = 32$

$4\sqrt{2}$

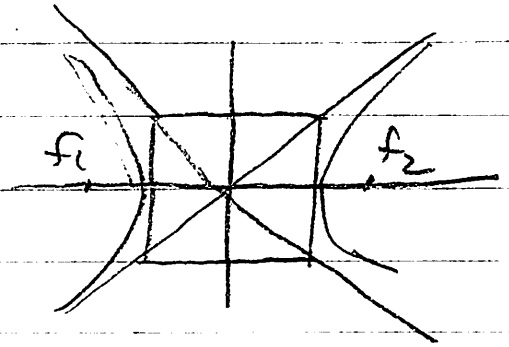
foci $(0, 4\sqrt{2}), (0, -4\sqrt{2})$

3.



$\frac{x^2}{16} + \frac{y^2}{25} = 1$

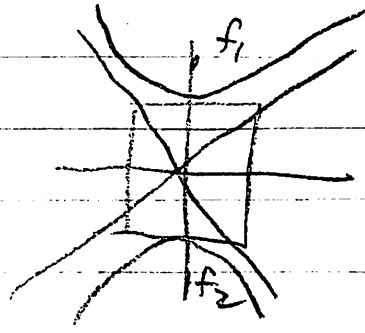
4. $x^2 - y^2 = 1$



$c^2 = a^2 + b^2$

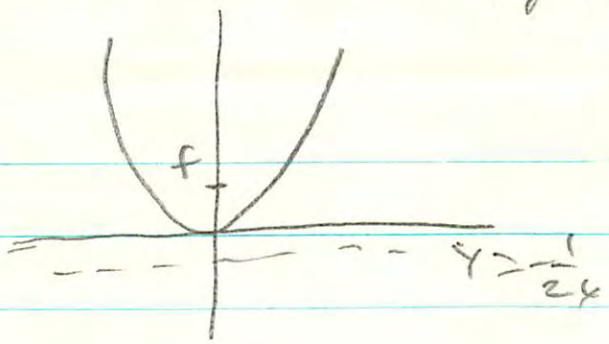
foci $(0, -\sqrt{2}), (0, \sqrt{2})$

5. $y^2 - x^2 = 1$



$(\sqrt{2}, 0), (-\sqrt{2}, 0)$

(6) $y = 6x^2$



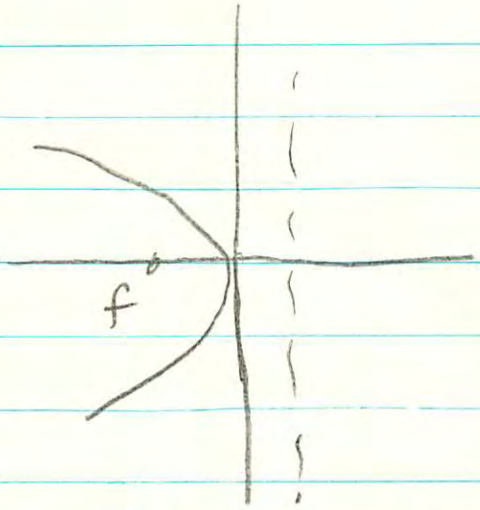
$\frac{1}{6}y = x^2$

$2c = \frac{1}{6}$

$c = \frac{1}{12}$ focus $0, \frac{1}{24}$

directrix $-\frac{1}{24}$

(7) $x = -6y^2$



$-\frac{1}{6}x = y^2$

$4cx = y^2$

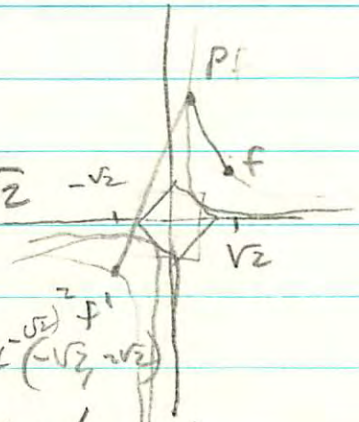
$2c = -\frac{1}{6}$

$c = -\frac{1}{12}$

$\left(-\frac{1}{24}, 0\right)$ focus

$x = \frac{1}{24}$ directrix

(8) $2a = 2\sqrt{2}$



$\sqrt{(x+\sqrt{2})^2 + (y+\sqrt{2})^2} - \sqrt{(x-\sqrt{2})^2 + (y-\sqrt{2})^2} = 2\sqrt{2}$

$\sqrt{\quad} = 2\sqrt{2} + \sqrt{\quad}$

$(x+\sqrt{2})^2 + (y+\sqrt{2})^2 = 8 + 4\sqrt{2}\sqrt{\quad} + (x-\sqrt{2})^2$

$x^2 + 2\sqrt{2}x + 2 + y^2 + 2\sqrt{2}y + 2 = 8 + 4\sqrt{2}(\quad) + x^2 - 2\sqrt{2}x + 2 + y^2 - 2\sqrt{2}y + 2$

$+2\sqrt{2}x + 2\sqrt{2}y = 8 + 4\sqrt{2}(\quad) - 2\sqrt{2}x - 2\sqrt{2}y$

$4\sqrt{2}x + 4\sqrt{2}y - 8 = +4\sqrt{2}\sqrt{\quad}$

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$$

$$\sqrt{\quad} = 2a - \sqrt{\quad}$$

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{\quad} + (x-c)^2 + y^2$$

$$x^2+2cx+c^2+y^2 = 4a^2 - 4a\sqrt{\quad} + x^2-2cx+c^2+y^2$$

$$4cx - 4a^2 = -4a\sqrt{(x-c)^2+y^2}$$

$$cx - a^2 = -a\sqrt{(x-c)^2+y^2}$$

$$a^2 - cx = a\sqrt{(x-c)^2+y^2}$$

$$a^4 - 2a^2cx + c^2x^2 = a^2(x^2 - 2cx + c^2 + y^2)$$

$$a^4 - 2a^2cx + c^2x^2 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$

$$a^4 + c^2x^2 = a^2x^2 + a^2c^2 + a^2y^2$$

$$a^2(a^2-c^2) = (a^2-c^2)x^2 + a^2y^2$$

$$1 = \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2}$$

9

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a$$

$$\sqrt{(x-c)^2+y^2} = \pm 2a + \sqrt{(x+c)^2+y^2}$$

$$(x-c)^2+y^2 = 4a^2 + 4a\sqrt{(x+c)^2+y^2} + (x+c)^2 + y^2$$

$$x^2 - 2cx + c^2 = \quad + x^2 + 2cx + c^2$$

$$-4cx = 4a^2 + 4a\sqrt{(x+c)^2+y^2}$$

$$-(cx + a^2) = a\sqrt{\quad}$$

$$c^2x^2 + 2a^2cx + a^4 = a^2\left[x^2 + 2cx + c^2 + y^2\right]$$

$$= a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2$$

$$c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

$$(c^2 - a^2)x^2$$

$$a^4 = (a^2-c^2)x^2 + a^2c^2 + a^2y^2$$

$$a^2(a^2-c^2) = (a^2-c^2)x^2 + a^2y^2$$

$$1 = \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2}$$

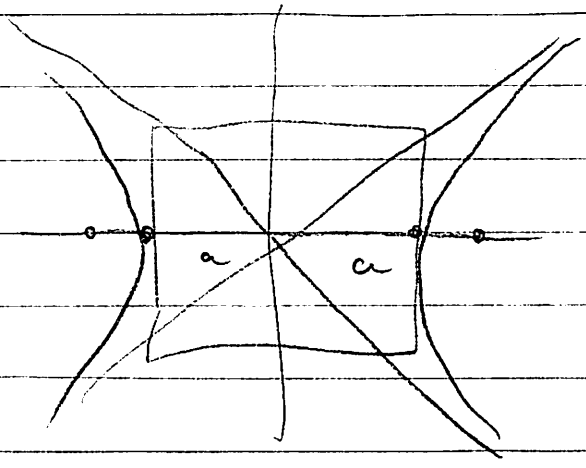
(10) $\sqrt{\left(x-\frac{c}{2}\right)^2+y^2} = \left(x+\frac{c}{2}\right)$

$x^2 - cx + \frac{c^2}{4} + y^2 = x^2 + cx + \frac{c^2}{4}$

$y^2 = 2cx$

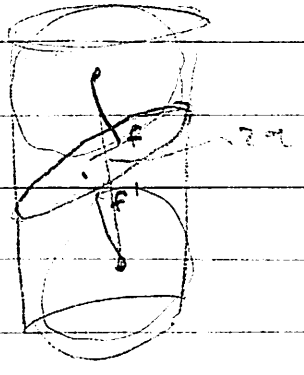
(11) $\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$

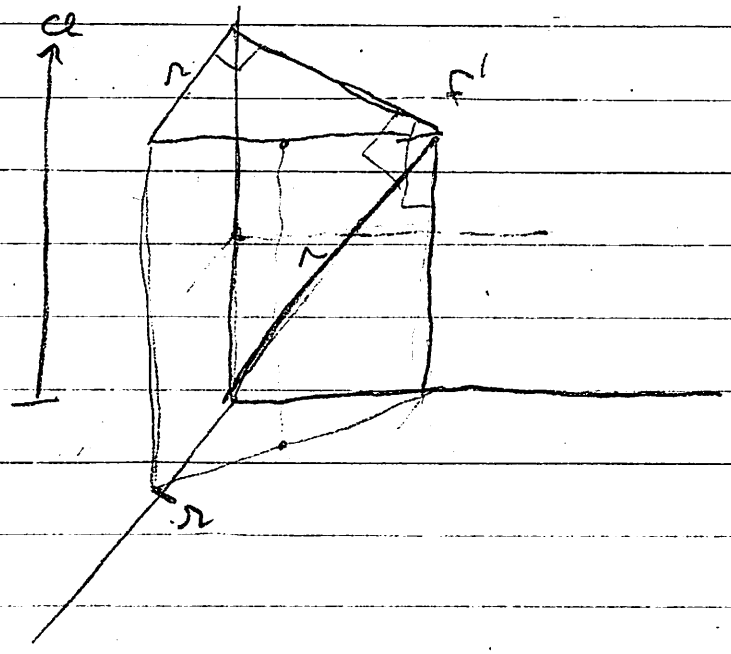
$\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$

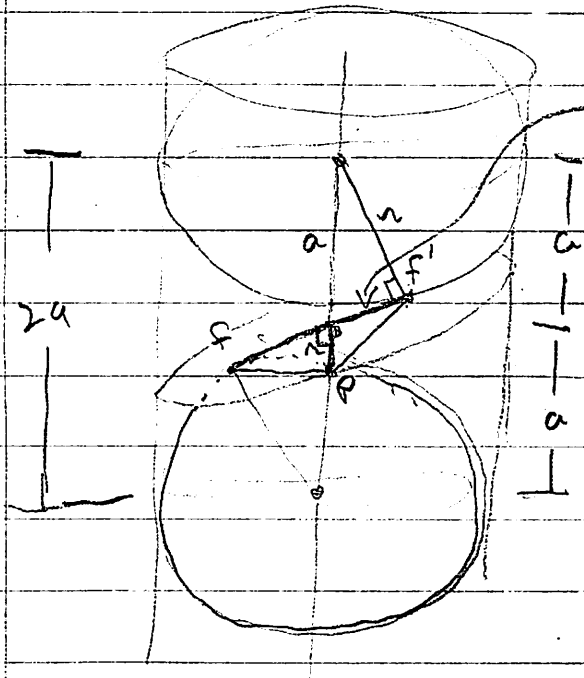


If $a^2 > c^2$ then equation looks like ellipse.
if b^2 is Negative.

(12)



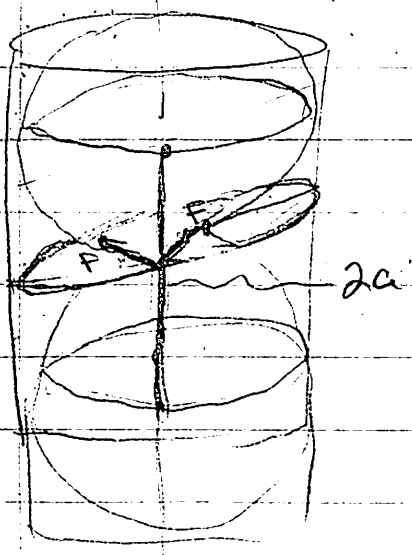


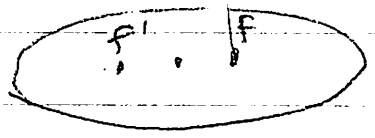


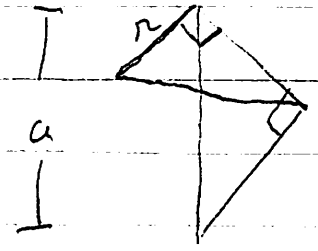
$$f'P = r^2 + \left(\sqrt{a^2-r^2}\right)^2 = a^2$$

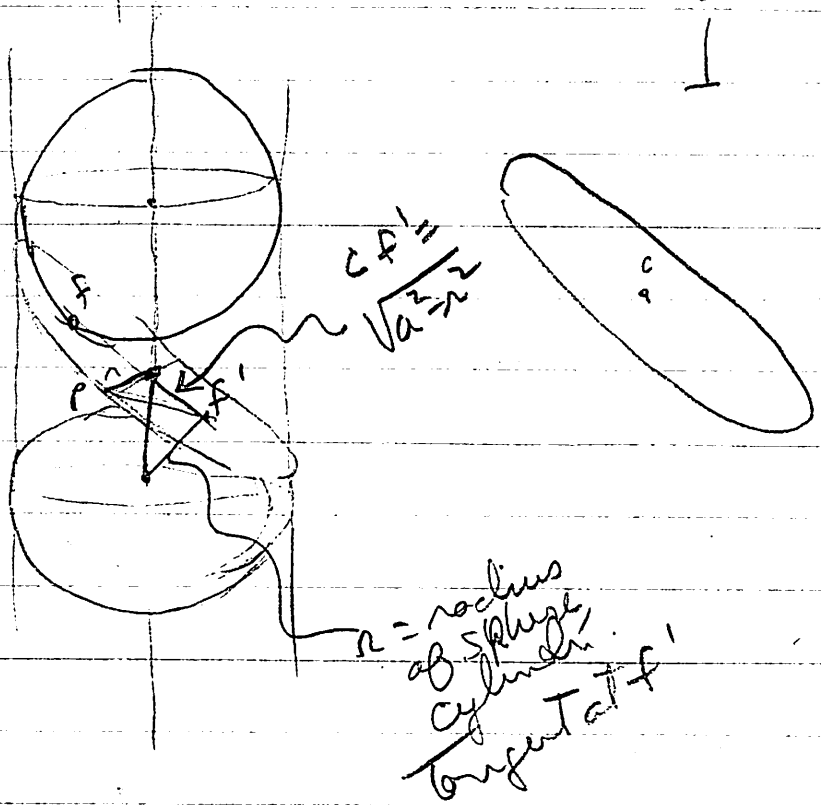
#12

Pg 9 20









minor axis length $r+r$

Point P on minor axis is radius of cylinder, which is r because PC is $\perp$ to axis of cylinder $\therefore$ = radius of cylinder.

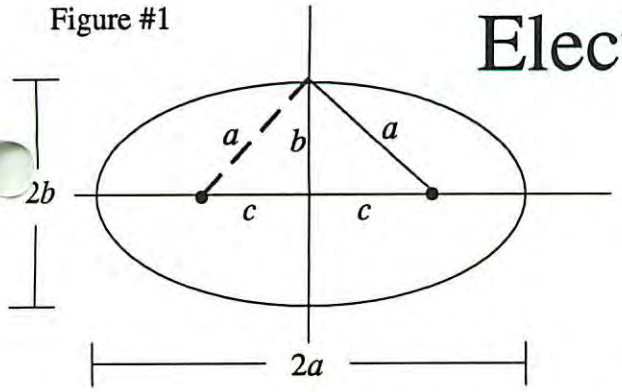
So $f'P = r^2 + \left(\sqrt{a^2-r^2}\right)^2 = a$

$+ \quad fP = r^2 + \left(\sqrt{a^2-r^2}\right) = a$

$= 2a$

Electric Eccentricity

Figure #1

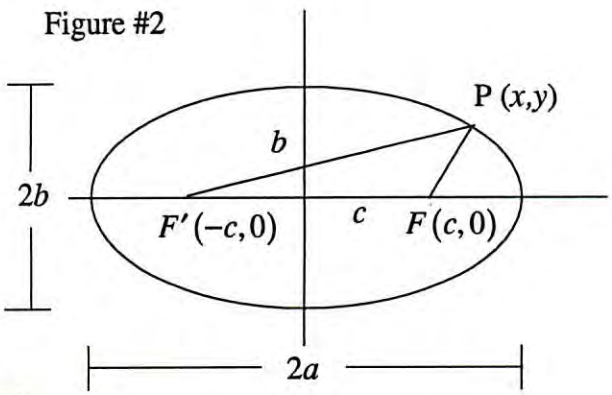


$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Algebra definition of an ellipse: $PF' + PF = 2a$

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

Figure #2



Isolate one radical and square

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$$

$$4cx=4a^2-4a\sqrt{(x-c)^2+y^2}$$

$$cx=a^2-a\sqrt{(x-c)^2+y^2}$$

$$a^2-cx=a\sqrt{(x-c)^2+y^2}$$

$$a^2-cx=a\underbrace{\sqrt{(x-c)^2+y^2}}_{PF}$$

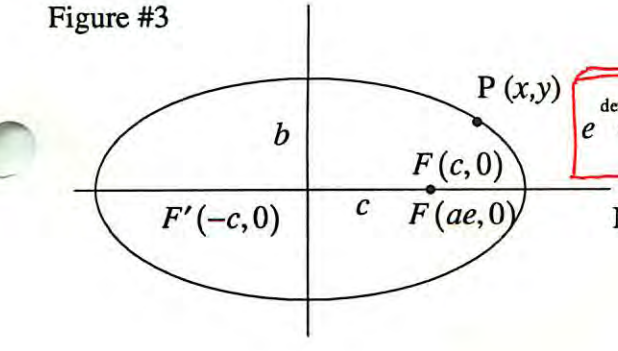
Solve for PF

$$PF=a-\frac{c}{a}x$$

This ratio is our main concern. Notice that $c < a$ because (see figure #1) the leg, c of the right triangle must be less than the hypotenuse, a so our ratio is less than 1 but greater than zero.

We shall call that ratio e.
e stands for the word *eccentricity* which refers to "how curvey" a line is *and **not** for the base of the natural logs.* You usually hear the word eccentric in reference to a person who acts differently than the norm. They are said to be "slightly off center." In our application the foci are "slightly off the center."

Figure #3



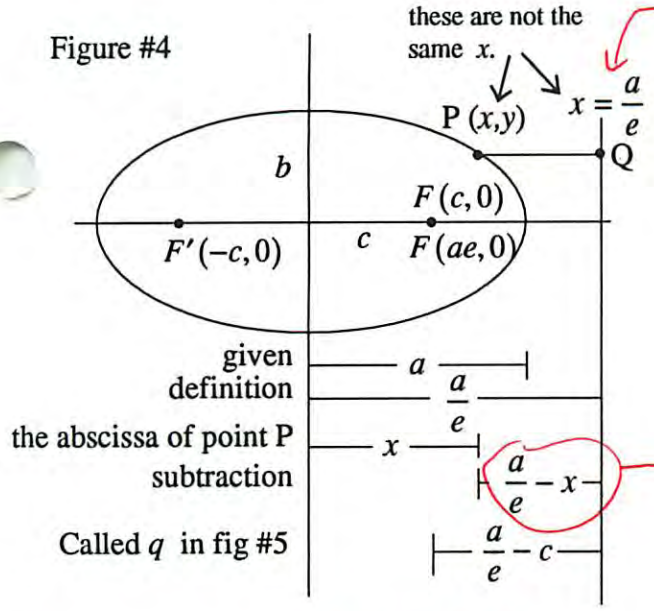
$e \overset{\text{defined}}{=} \frac{c}{a}$ so $c = ae$

Notice: $c < a$ so $e < 1$ but positive.

Thus, $PF = a - ex$
factor e

$$PF=e\left(\frac{a}{e}-x\right)$$

Figure #4



This is the equation of a line.

What can we say about this equation? $PF = e\left(\frac{a}{e} - x\right)$

We know e is less than 1 so $\frac{a}{e} > a$ because division by a number smaller than 1 yields a larger quotient (Thus, $\frac{a}{e}$ is beyond the right edge of the ellipse which is a from the center. Let's define the vertical line (parallel to the y-axis) drawn through $\frac{a}{e}$ to be the directrix $x = \frac{a}{e}$. Now, $PF = e\left(\frac{a}{e} - x\right)$ becomes $PF = ePQ$ (where PQ is the distance from P to the directrix.) which leads to:

Remember, e defined $\frac{c}{a}$

$$e = \frac{PF}{PQ} = \frac{\text{distance from focus to P}}{\text{distance from P to directrix}} = \frac{c}{a}$$

Figure #5

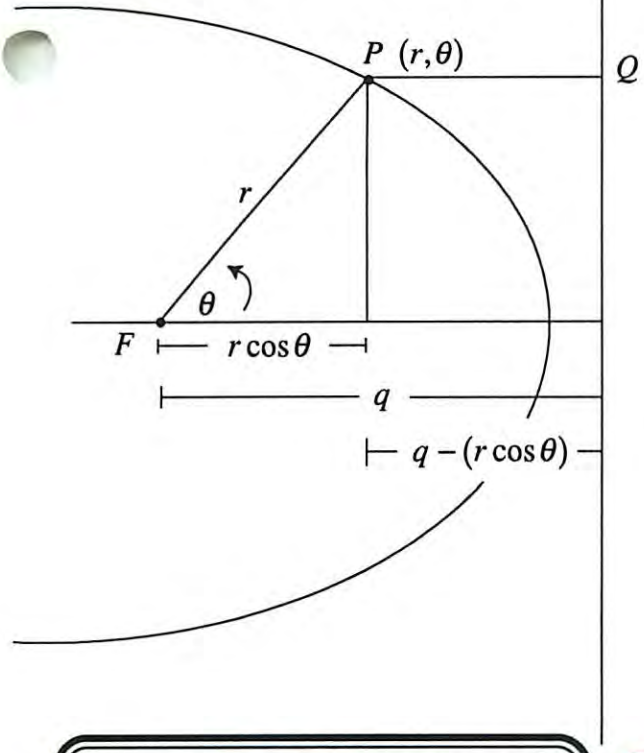


Figure #5 is a blow-up of a portion of figure #4.

$$e = \frac{PF}{PQ} = \frac{r}{q - (r\cos\theta)}$$

now, solve for r

$$eq - er\cos\theta = r$$

$$eq = r + er\cos\theta$$

$$r(1 + e\cos\theta) = eq$$

$$r = \frac{eq}{1 + e\cos\theta}$$

Depending on the orientation of the conic, one of the following is the actual "general case" ***if and only if a focus is located at the pole.***

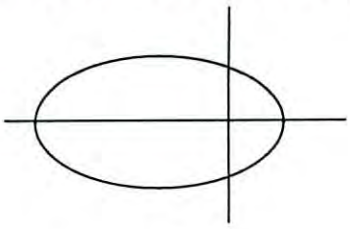
$$r = \frac{eq}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{eq}{1 \pm e\sin\theta}$$

"Orientation" refers to the relative location of the foci. In the illustrated example, the right-hand focus was at the pole *and* the major axis was horizontal

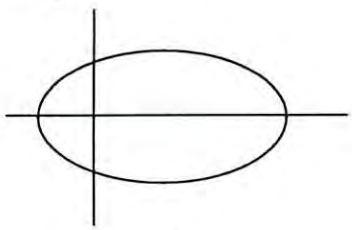
Furthermore, ~~However~~, regardless of orientation, the following is true:

If $e = 0$ then the conic is a circle.
If $e < 1$ then the conic is an ellipse.
If $e = 1$ then the conic is a parabola.
If $e > 1$ then the conic is a hyperbola.

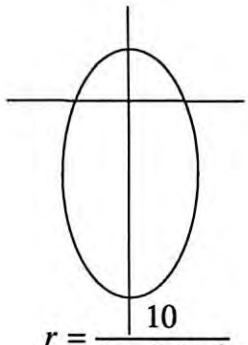
Electric Eccentricity



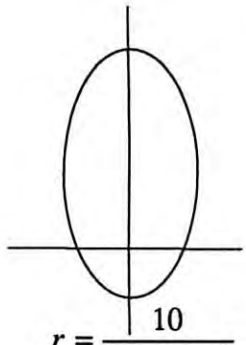
$$r = \frac{10}{3 + 2\cos\theta}$$



$$r = \frac{10}{3 - 2\cos\theta}$$



$$r = \frac{10}{3 + 2\sin\theta}$$



$$r = \frac{10}{3 - 2\sin\theta}$$

See figure # 6 below.

Sketch : $r = \dfrac{10}{3 + 2\cos\theta}$

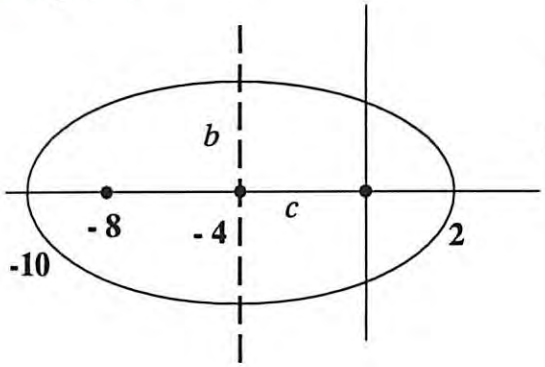
When $\theta = 0$ $r = \dfrac{10}{5} = 2$ and when $\theta = 180$ $r = \dfrac{10}{3-2} = 10$. 2 along the positive x-axis and 10 (actually –10) along the negative x-axis. These are end points of the figure. The line has a length of 2 –(–10)=12.

When $\theta \in \{90,\ 270\}$ $r = \dfrac{10}{3}$. Remember, this is vertical distance *from the pole*!

$$r = \frac{\frac{10}{3}}{\frac{3}{3} + \frac{2}{3}\cos\theta}$$

Because of the $\frac{2}{3}$, we know we are working with an ellipse. The major axis is horizontal and the minor axis is vertical (because we have cosine instead of sine in our equation). The foci are located on the major axis. "a" is equal to one-half of the length of the major axis, so $a = 6$. The center of the ellipse: $2 - 6 = -4$. (4 on the negative x-axis).

Figure #6



$\frac{c}{a} = e$ which becomes $\frac{c}{6} = \frac{2}{3}$

So, $c = 4$ and $a^2 = b^2 + c^2$ becomes $6^2 = b^2 + 4^2$; thus $b = 2\sqrt{5}$

from figure #1 on the first page

$$r = \frac{\frac{2}{3} \cdot 5}{\frac{3}{3} + \frac{2}{3}\cos\theta}$$

5 is the distance from the a focus to its directrix. So right hand directrix is $x = 5$ and the left hand directrix is $x = -13$ (5 left of left focus).

$$r = \frac{10}{3 + 2\cos\theta}$$

$$3r + 2r\cos\theta = 10$$

$$3\sqrt{x^2 + y^2} + 2x = 10$$

$$3\sqrt{x^2 + y^2} = 10 - 2x$$

$$9x^2 + 9y^2 = 100 - 40x + 4x^2$$

$$5x^2 + 40x + 9y^2 = 100$$

$$5\left(x^2 + 8x + 16\right) + 9y^2 = 100 + 80$$

$$5(x+4)^2 + 9y^2 = 180$$

$$\frac{(x+4)^2}{36} + \frac{y^2}{20} = 1$$

$$a = 6 \qquad b = 2\sqrt{5}$$

Electric Eccentricity

Convert to polar notation:

$$\frac{(x \oplus 4)^2}{36} + \frac{y^2}{20} = 1$$

These signs agree

$$r = \frac{}{\oplus \cos}$$

1) Given is graphing form of an ellipse —> + and equals 1.

2) Major axis horizontal $(36 > 20)$ so we will have cosine in our equation instead of sine

3) $a = 6 \quad b = 2\sqrt{5} \quad \therefore \quad c = \sqrt{36 - 20} = 4$

4) $e = \frac{c}{a} = \frac{4}{6} = \frac{2}{3}$ ⟶ $r = \frac{\frac{2}{3}\ ?}{1 + \frac{2}{3}\cos\theta}$

5) The question mark is the distance from the focus to the directrix. See figure # 4

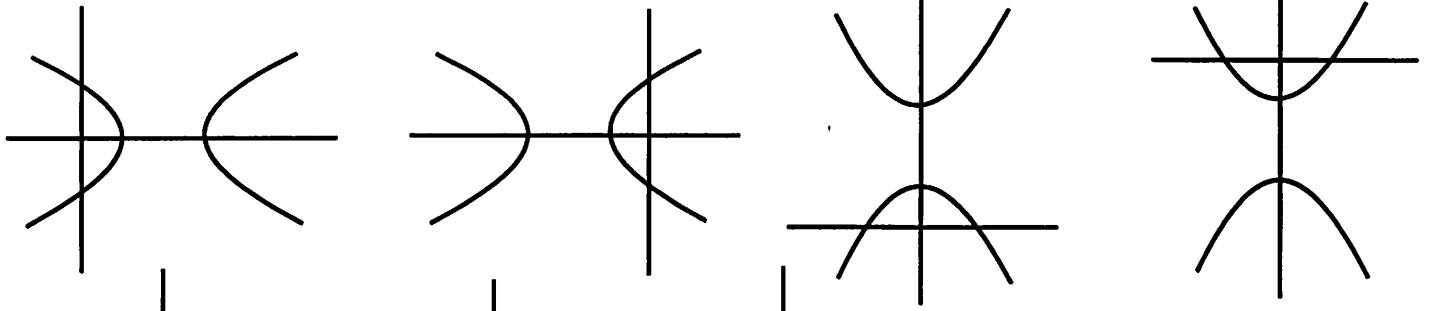
6) The distance from the center of the ellipse to the directrix is: $\frac{a}{e} = \frac{a}{\frac{c}{a}} = \frac{a^2}{c} = \frac{36}{4} = 9.$

So, from $x = -4$ to the right directrix is $-4 + 9 = 5$ and $-4 - 9 = -13$. That is 13 to left directrix.

7) See fig #4, $\frac{a}{e} - c = \frac{6}{\frac{2}{3}} - 4 = 6 \cdot \frac{3}{2} - 4 = 9 - 4 = \textcircled{5}$

$r = \frac{\frac{2}{3} \cdot \textcircled{5}}{1 + \frac{2}{3}\cos\theta}$ $\qquad$ $r = \frac{\frac{2}{3} \cdot 5}{1 + \frac{2}{3}\cos\theta} \cdot \frac{3}{3}$ $\qquad$ $r = \frac{10}{3 + 2\cos\theta}$

$r = \frac{10}{2 + 3\cos\theta}$ $\qquad$ $r = \frac{10}{2 - 3\cos\theta}$ $\qquad$ $r = \frac{10}{2 + 3\sin\theta}$ $\qquad$ $r = \frac{10}{2 - 3\sin\theta}$



$r = \frac{10}{3 + 3\cos\theta}$ $\qquad$ $r = \frac{10}{3 - 3\cos\theta}$ $\qquad$ $r = \frac{10}{3 + 3\sin\theta}$ $\qquad$ $r = \frac{10}{3 - 3\sin\theta}$

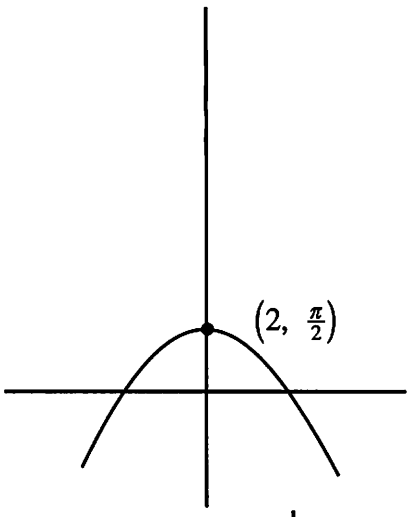
Electric Eccentricity

$$r = \frac{10}{2+3\sin\theta} = \frac{5}{1+\frac{3}{2}\sin\theta}$$

$e = \frac{3}{2} > 1 \quad \therefore$ Hyperbola

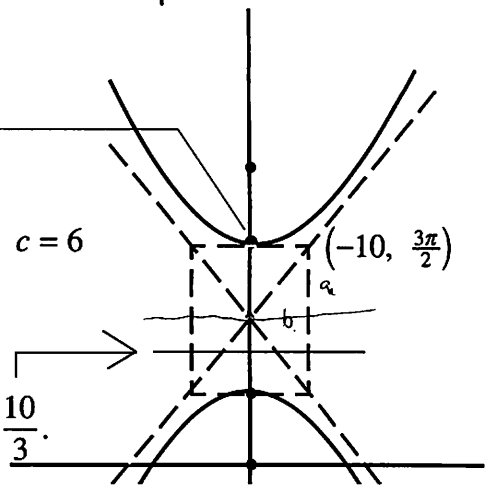
Transverse axis (the one containing the foci) is vertical (because of sine).
The focus is at the lower pole because the equation is in the form:

$$r = \frac{eq}{1+e\sin\theta}$$



When $\theta = \frac{\pi}{2} \longrightarrow r = 2$

When $\theta = \frac{3\pi}{2} \longrightarrow r = -10$



The length of the transverse axis is $10-2=8$ so $a=4$ and $e = \frac{3}{2} = \frac{c}{a} = \frac{c}{4} \quad \therefore \quad c = 6$

$b^2 = c^2 - a^2 = 36 - 16 = 20 \quad \therefore \quad b = 2\sqrt{5}$

$r = \frac{eq}{1+e\sin\theta} = \frac{5}{1+\frac{3}{2}\sin\theta} = \frac{\frac{3}{2}\cdot\frac{10}{3}}{1+\frac{3}{2}\sin\theta}$ so distance from focus to its directrix is $\frac{10}{3}$.

The distance from the "center" to the directrix is $\frac{a}{e}$ exactly as it is in the ellipse however the distance from the focus to the directrix is $q = c - \frac{a}{e}$ The negative of what it is for the ellipse. This is accomplished because e is greater than 1 in a hyperbola

---

I Identify, sketch ,and find the rectangular equivalent and graph of the given equation. Specifically state the equation(s), etc. of asymptotes, directrix, foci, vertex, etc.

a) $r = \frac{6}{3+\sin\theta}$ b) $r = \frac{6}{1-3\sin\theta}$ c) $r = \frac{3}{2+2\cos\theta}$

II Express the given equation in polar form and then find the eccentricity and an equation for the directrix:

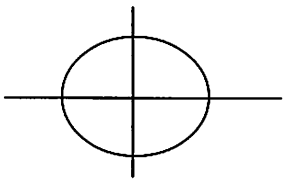
a) $y^2 = 4 - 4x.$ b) $3y^2 - x^2 - 12y + 9 = 0$

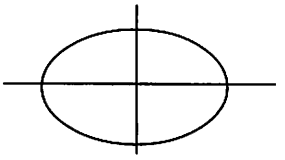
c) $3x^2 + 4y^2 + 4x - 4 = 0$ d) make up one of your own

III Match each ellipse with its equation:

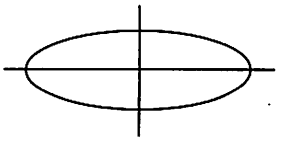
a) $r = \frac{10}{5+2\cos\theta}$ b) $r = \frac{10}{5+3\cos\theta}$

c) $r = \frac{10}{5+4\cos\theta}$ d) conclusion ...



1) 

2)

3) 

I

a) $r = \dfrac{12}{6+2\sin\theta}$

$= \dfrac{\textcircled{1}\ \textcircled{2}}{1+\frac{1}{3}\sin\theta}$

(2) $e = \frac{1}{3}$ (3) $\frac{1}{3}\cdot 6 = 2$

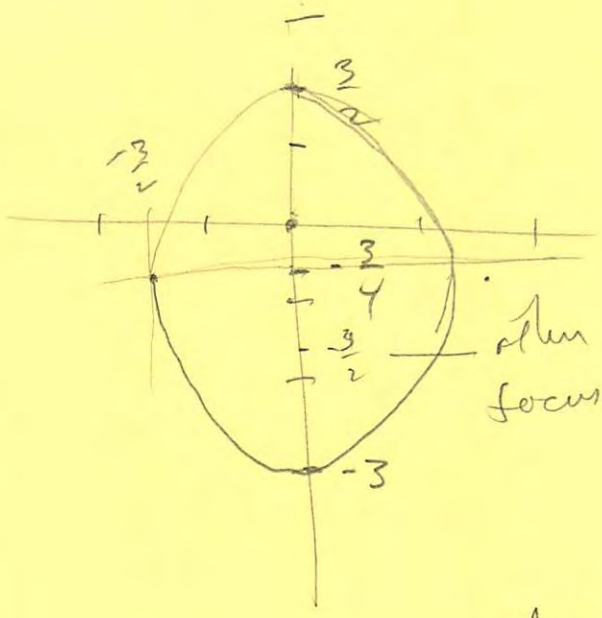
(4) $\therefore$ ellipse — dist fm focus to directrix

(5) "sin" major axis is vertical

(6) "+" upper focus at Pole.

$\theta = 90 \quad \frac{12}{8} = \frac{3}{2}$

$\theta = 270 \quad \frac{12}{4} = 3$



other focus $\frac{3}{4}$ below center is $-\frac{3}{2}$

$3 + \frac{3}{2} = \frac{9}{2} \div 2 = \frac{9}{4}$ $\quad 3 - \frac{9}{4} = \frac{-3}{4}$ is center of ellipse

$\frac{9}{4}$ ← length of major axis; "b"

Dist fm center to focus is "c". focus is at pole

So $c = \frac{3}{4}$

$b^2 = c^2 + a^2$

$\frac{81}{16} = \frac{9}{16} + a^2$

$\frac{72}{16} = a^2$

$\frac{9}{2} = a^2$

$a = \frac{3\sqrt{2}}{2}$ $\quad \frac{3}{\sqrt{2}}$

$\frac{b}{e} = \frac{9}{4}\cdot 3 = \frac{27}{4} = 6\frac{3}{4}$

center $(0, -\frac{3}{4})$

$\frac{x^2}{\frac{9}{2}} + \frac{\left(y+\frac{3}{4}\right)^2}{\left(\frac{9}{4}\right)^2} = 1$

equation of directrix $y = 6$. upper. $\frac{81}{16}$

other focus $-\frac{3}{2} - 6 = -\frac{15}{2} = 4$ is other directrix

$\left(\frac{x^2}{\frac{9}{2}} + \frac{\left(y+\frac{3}{4}\right)^2}{\frac{81}{16}} = 1\right)\frac{81}{16}$

$\frac{9}{8}x^2 + \left(y+\frac{3}{4}\right)^2 = \frac{81}{16}$

$9x^2 + 8\left(y+\frac{3}{4}\right)^2 = \frac{81}{2}$

I A

$$r = \frac{6}{3 + \sin\theta}$$

$$3r + r\sin\theta = 6$$

$$3\sqrt{x^2+y^2} + y = 6$$

$$3\sqrt{x^2+y^2} = 6 - y$$

$$9x^2 + 9y^2 = 36 - 12y + y^2$$

$$9x^2 + 8y^2 + 12y - 36 = 0$$

$$9x^2 + 8\left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = 36 + \frac{9}{2}$$

$$9x^2 + 8\left(y + \frac{3}{4}\right)^2 = \frac{81}{2}$$

$$\frac{x^2}{8} + \frac{\left(y+\frac{3}{4}\right)^2}{9} = \frac{9}{16}$$

$$\frac{x^2}{\frac{9}{2}} + \frac{\left(y+\frac{3}{4}\right)^2}{\frac{81}{16}} = 1$$

$$\frac{x^2}{\frac{3\sqrt{2}}{2}} + \frac{\left(y+\frac{3}{4}\right)^2}{\left(\frac{9}{4}\right)^2} = 1$$

c) $r = \frac{4}{\cos\theta - 2}$ Just an ellipse.

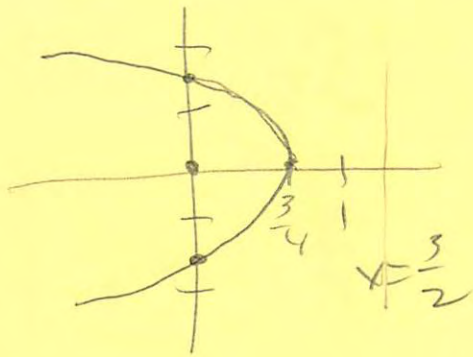
$$= \frac{4}{-2 + \cos\theta}$$

$$= \frac{-2}{1 - \frac{1}{2}\cos\theta}$$

I

6) $r = \dfrac{3}{2+2\cos\theta}$

$= \dfrac{\frac{3}{2}}{1+\cos\theta}$



$e = 1$ dist fm focus to dir is $\frac{3}{2}$

$\therefore$ Parabola vertex is $\frac{1}{2}$ way between.

$\theta = 0$

$= 90$

$= 270$

$2r + 2r\cos\theta = 3$

$2(x^2+y^2)^{\frac{1}{2}} + 2x = 3$

$4x^2 + 4y^2 = 4x^2 - 12x + 9$

$x = \frac{-1}{3}y^2 + \frac{3}{4}$

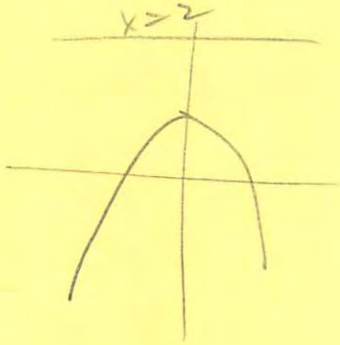
c $r = \dfrac{3}{1+2\sin\theta}$ $\quad \dfrac{-x^2}{3} + \dfrac{(y-2)^2}{1} = 1$

~~X~~ B $\left(\frac{3}{2}, 2\right)$

$-x^2 + 3(y^2 - 4y + 4) = 3$

$3y^2 - x^2 - 12y + 9 = 0$

D $r = \dfrac{2}{1+\sin\theta}$ $\quad x^2 + 4y - 4 = 0$



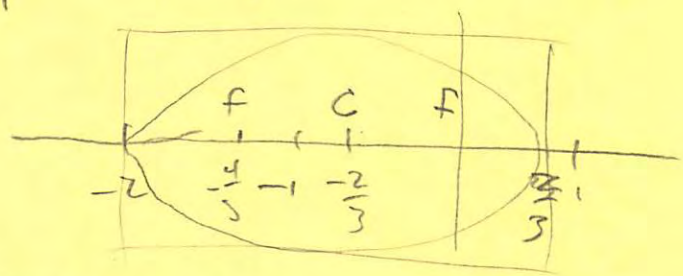
~~II~~ V

B $r = \dfrac{1}{1+\frac{1}{2}\cos\theta}$ $\quad \dfrac{(x+\frac{2}{3})^2}{\frac{16}{9}} + \dfrac{y^2}{\frac{4}{3}} = 1$

$x^2 + \frac{4}{3}x + \frac{4}{9} + \frac{4}{2}y^2 = \frac{16}{9}$

$x^2 + \frac{4}{3}y^2 + \frac{4}{3}x - \frac{14}{3} = 0$

$3x^2 + 4y^2 + 4x - 4 = 0$



C $r = \dfrac{4}{1-2\cos\theta}$ $\quad \dfrac{9\left(x-\frac{8}{3}\right)^2}{16} - \dfrac{3y^2}{16} = 1$

$9\left(x^2 - \frac{16}{3}x + \frac{64}{9}\right) - 3y^2 = 16$

$9x^2 - 3y^2 - 16x + 48 = 0$

---

f) $r = \dfrac{3}{1-2\cos\theta}$

dist f'to P dist f to P

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

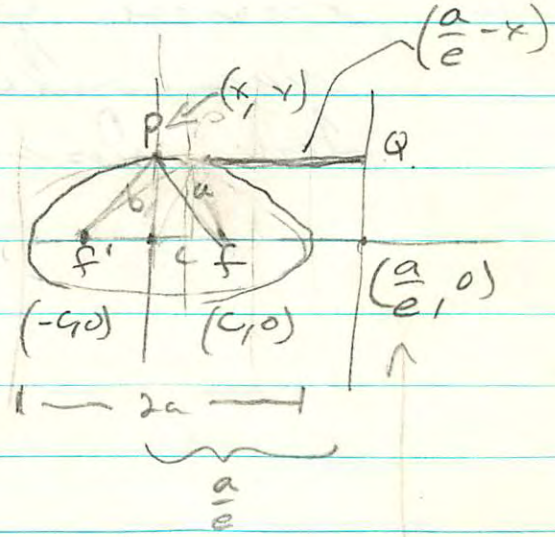
Begin again not proper

$$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$$

$$x^2+2cx+c^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2-2cx+c^2+y^2$$

$$-4cx = 4a^2 - 4a\sqrt{(x-c)^2+y^2}$$

$$a^2 - cx = a\underbrace{\sqrt{(x-c)^2+y^2}}_{\overline{Pf}}$$



$$cx - a^2 = a\,\overline{Pf}$$

$$\overline{Pf} = a - \frac{cx}{a}$$

$$= a - \frac{c}{a}x \qquad \text{define } \frac{c}{a} = e = \text{eccentricity}$$

$$= a - ex$$

$x = \frac{a}{e}$ is equation of this "directrix"

$$\overline{PF} = e\underbrace{\left(\frac{a}{e} - x\right)}_{PQ}$$

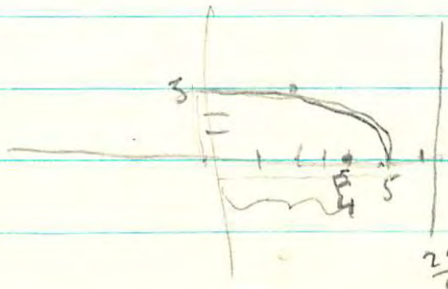
$$\overline{PF} = e\,\overline{PQ}$$

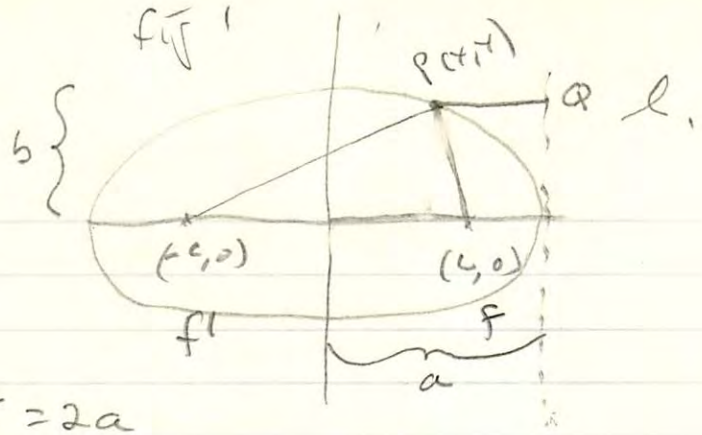
$$\frac{\overline{PF}}{PQ} = e \quad \text{is a constant}$$

eg 1 Pg 922 $\frac{x^2}{25} + \frac{y^2}{9} = 1$ $a=5$, $b=3$ $e = \frac{c}{a}$ $c = a^2 - b^2$

$e = \frac{4}{5}$ ∴ this is ellipse. ∴ $c = 4$

$x = \frac{5}{\frac{4}{5}} = \frac{25}{4}$





Def of ellipse $\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$

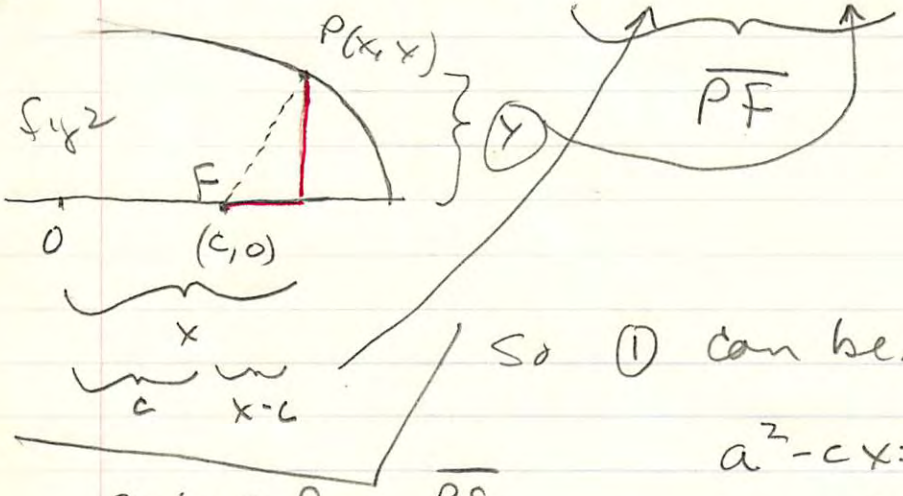
$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2$

$x^2+2cx+c^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2$

$2cx = 4a^2 - 4a\sqrt{(x-c)^2+y^2} - 2cx$

$4cx = 4a^2 - 4a\sqrt{(x-c)^2+y^2}$

① $a^2 - cx = a\sqrt{(x-c)^2+y^2}$



So ① can be rewritten

$a^2 - cx = a\overline{PF}$

Solve for $\overline{Pf}$

$\overline{Pf} = \frac{a^2-cx}{a} = a - \frac{c}{a}x$

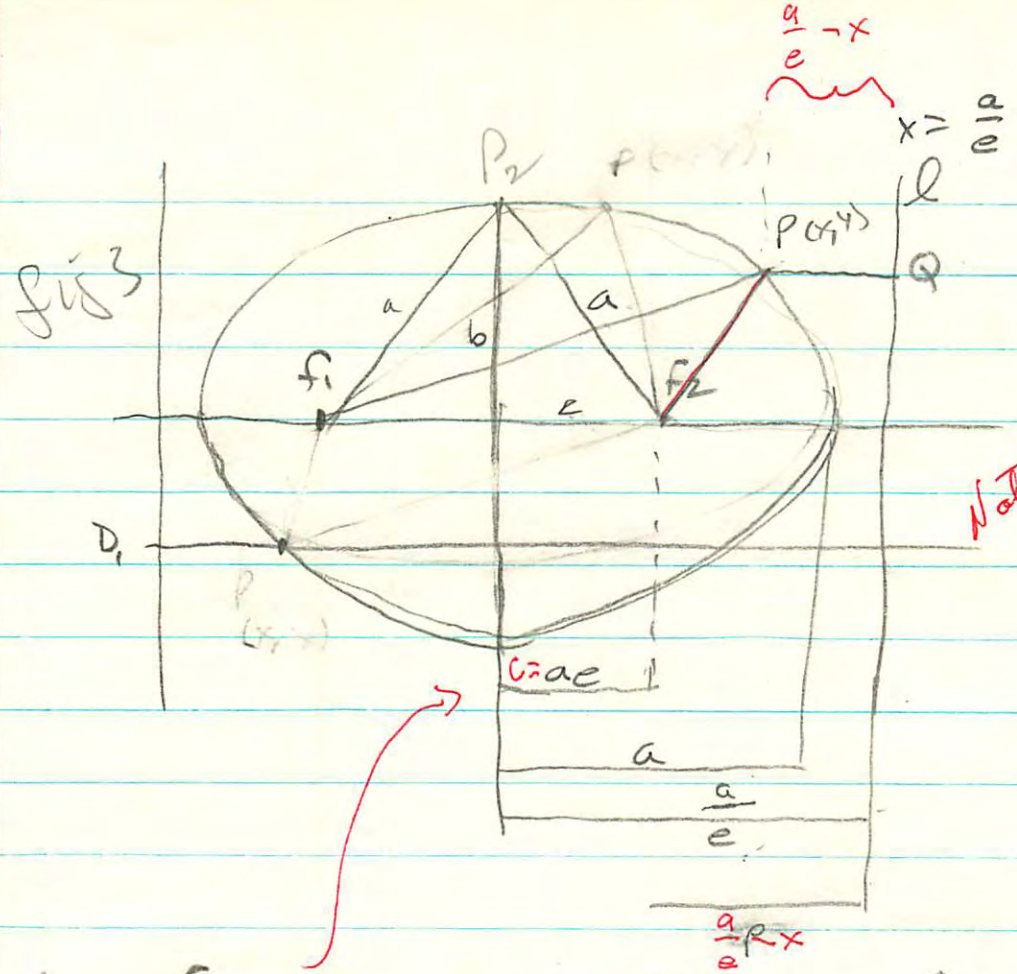
let $\frac{c}{a} = e$ Then $\overline{Pf} = a - ex$

also $c = ae$ $\overline{Pf} = e\left(\frac{a}{e} - x\right)$

PQ see fyj 3

$\overline{Pf} = e\,\overline{PQ}$

$\frac{Pf}{PQ} = e$ a constant!



Pg 921 $\quad \frac{c}{a} = e$

$\overline{Pf} = a - \frac{c}{a}x$

$\overline{Pf} = a - ex$

(Pg 922) $\quad \frac{\overline{PF}}{\overline{PQ}} = e$

when $c = 0$, f is at center of circle.

as $c \to a$ ellipse flattens.

$\frac{a - ex}{e} = PQ$

$\frac{a}{e} - x = PQ$

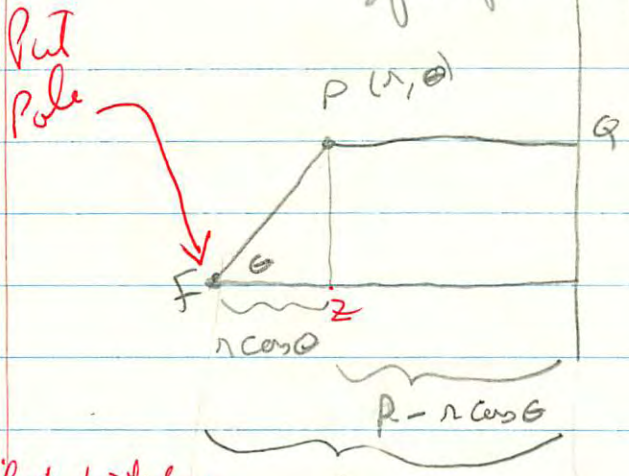
use P_2 to compute the idea $b^2 + c^2 = a^2$ — in ellipse

also note $c < a$ — so ratio of $\frac{c}{a} < 1$

leg < HYP

Developing Polar form of equation Pg 923

using def of e.



$$e = \frac{\overline{FP}}{\overline{PQ}}.$$

$$e = \frac{r}{R - r\cos\theta}$$

$$er - re\cos\theta = r$$

$$er = r(1 + e\cos\theta)$$

$$r = \frac{eR}{1 + e\cos\theta}$$

① diff dist from F to l as p → P

② $\overline{FZ} = r\cos\theta$

③ Z to l = $R - r\cos\theta$.

eg 2 Pg 923

most books show the formula as

$$r = \frac{e(p)}{1 + e\cos(\theta - \beta)}$$

$$r = \frac{8}{5 + 6\cos\theta}$$

$$= \frac{\frac{8}{5}}{1 + \frac{6}{5}\cos\theta}$$

$$\frac{8}{5} = \frac{6}{5} \cdot x$$

we called it "a" in centuries

$$= \frac{\frac{8}{5} \cdot \frac{8}{6}}{1 + \frac{6}{5}\cos\theta}$$

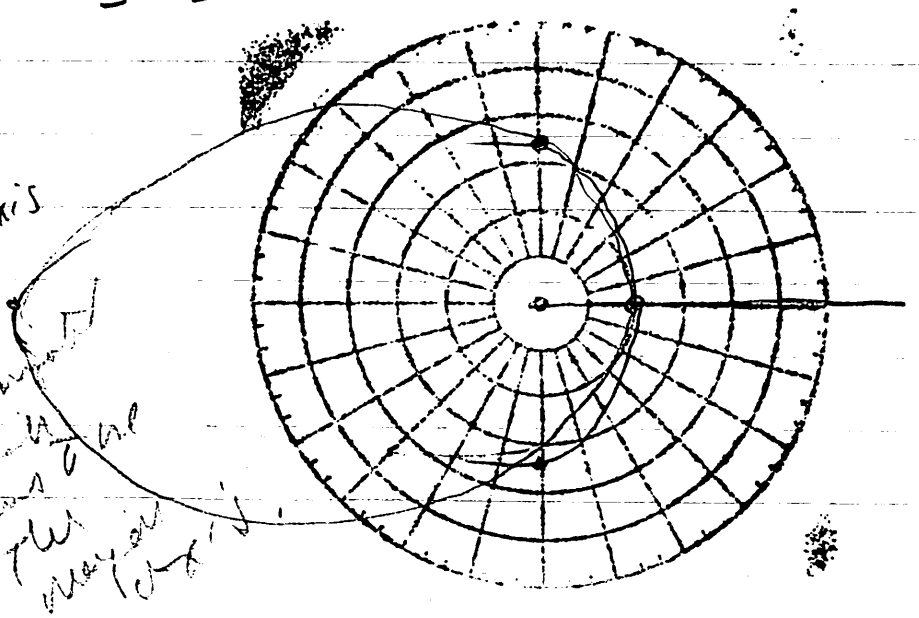
$$= \frac{\frac{8}{5} \cdot \left[\frac{4}{3}\right]}{1 + \left(\frac{6}{5}\right)\cos\theta}$$

dist from focus to directrix for ellipse

∴ Hyp.

① $r = \dfrac{10}{3+2\cos\theta}$

r	θ
2	0
$\frac{10}{3}$	$\frac{\pi}{2}$
10	π
$\frac{10}{3}$	$\frac{3\pi}{2}$



Note: used "cos" in formula, implies major axis is horizontal, had "sin" been used, major axis would be vertical, actually, ours is horizontal, cosine is in the denominator & the major axis is horizontal

$$r = \frac{ep}{1+e\cos(\theta-\beta)}$$

need to achieve this

$$3r + 2r\cos\theta = 10$$

$$3\sqrt{(x^2+y^2)} + 3x = 10$$

$$3\sqrt{x^2+y^2} = 10 - 2x$$

$$9x^2 + 9y^2 = 100 - 40x + 4x^2$$

$$5x^2 + 40x + 9y^2 = 100$$

$$5(x^2+8x+16) + 9y^2 = 100 + 80$$

$$r = \frac{\frac{10}{3}}{1+\frac{2}{3}\cos\theta}$$

since $\frac{2}{3}$ is less than 1, figure is an ellipse.

$$5(x+4)^2 + 9y^2 = 180$$

$$\frac{(x+4)^2}{36} + \frac{y^2}{20} = 1$$

3 a) $r = \dfrac{5}{3+4\cos\theta}$

$\dfrac{\frac{5}{3}}{1+\frac{4}{3}\cos\theta}$ $\quad e = \frac{4}{3}$ Hyperbola

b) $r = \dfrac{5}{4+3\cos\theta}$

$\dfrac{\frac{5}{4}}{1+\frac{3}{4}\cos\theta}$ $\quad e = \frac{3}{4}$ ellipse

c) $r = \dfrac{5}{3+3\cos\theta}$

$\dfrac{\frac{5}{3}}{1+\cos\theta}$ $\quad e = 1$ parabola

d) $r = \dfrac{5}{3-4\cos\theta}$ $\quad e = \frac{4}{3}$ Hyperbola

$= \dfrac{\frac{5}{3}}{1-\frac{4}{3}\cos\theta}$

also see comment on p1m 1.

The neg ↗ implies the directrix is to the left of the pole rather than to the right as in earlier problems.

General form $r = \dfrac{ep}{1+e\cos(\theta-\beta)}$ ↙ if $\beta = 180°$

$$r = \frac{ep}{1 + e[\cos\theta\cos\pi + 0]}$$

$$r = \frac{ep}{1 - e\cos\theta}$$

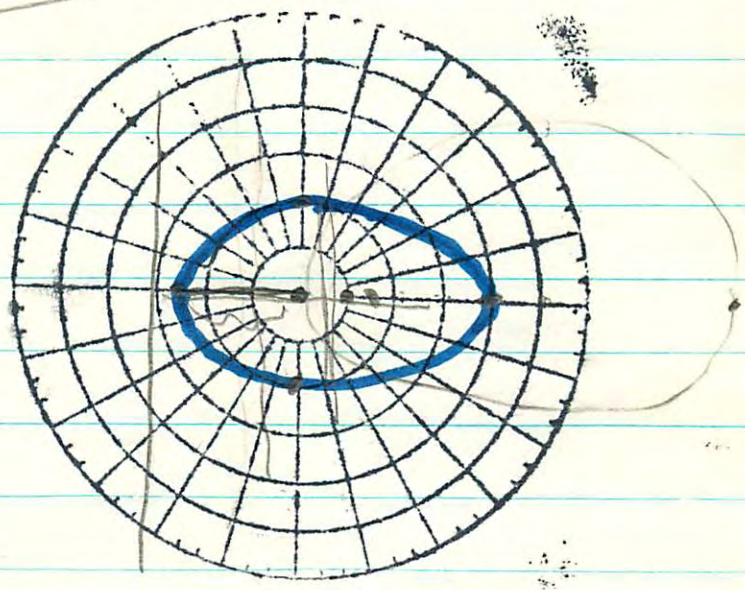
this gets us moved to the left hand focus ←

NO! in wrong form. Pg 923

4. a) $r = \dfrac{8}{1 - \frac{1}{2}\cos\theta}$ $\quad e = \frac{1}{2} < 1 \therefore$ ellipse

OK, it's alright.

$r = \dfrac{ep}{1 + e\cos(\theta - \beta)}$ $\quad \beta = \pi$



b)

θ	ρ
0	16
$\frac{\pi}{2}$	8
π	$\frac{16}{3}$
$\frac{3\pi}{2}$	8
2π	16

48
16
64/3 32/3

$a = \frac{32}{3}$

c $\quad \overline{Pf} = \frac{a}{e} - ex \qquad r = \dfrac{ep}{1 + e\cos\theta}$

r

$\frac{a}{e} \quad \frac{a}{\frac{1}{2}} = 2a$

$ep = er\cos(\theta - \beta) = r$

$ep = r + er\cos(\theta - \beta)$

$\rho = \frac{r}{e} + r\cos(\theta - \beta)$

dist to center is $\frac{1}{2}$ way between left & right "ends" $\dfrac{16 + \frac{16}{3}}{2} = \frac{32}{3}$.

This is a.

$\therefore \frac{8}{3} + \frac{8}{3} = \frac{24+8}{3} = \frac{32}{3}$

Continued

4 cont

$$r = \frac{8}{1 - \frac{1}{2}\cos\theta}$$

$\theta = 0 \quad r = 16$

$\theta = \pi \quad r = \frac{16}{3}$

$$\left(\frac{16}{3} + \frac{3 \cdot 16}{3}\right)\frac{1}{2}$$

$$\frac{8}{3} + \frac{24}{3}$$

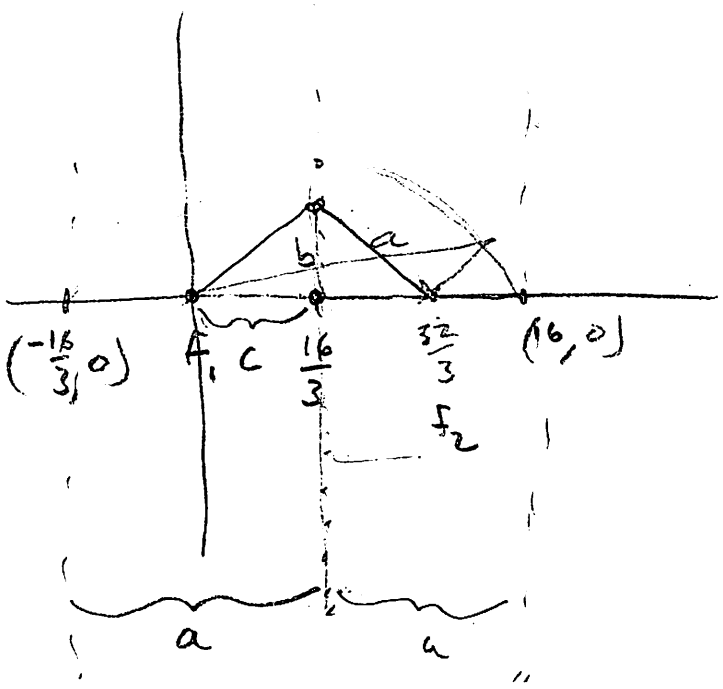
$$\frac{32}{3} = a$$

$$e = \frac{1}{2} = \frac{c}{a}$$

$$a = 2c$$

$$\frac{32}{3} = 2c$$

$$\frac{16}{3} = c$$



1st focus $(0, 0)$

2nd focus $(\frac{32}{3}, 0)$

$$a^2 = b^2 + c^2$$

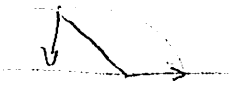
$$\frac{32^2}{3^2} = b^2 + \frac{16^2}{3^2}$$

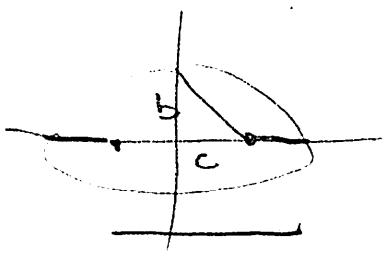
$$\frac{32 \cdot 32}{3 \cdot 3} - \frac{16 \cdot 16}{3 \cdot 3} = b^2$$

$$\frac{16^2}{3^2}\left[4 - 1\right] = b^2$$

$$\frac{16^2}{3} = b^2$$

$$b = \frac{16}{\sqrt{3}}$$





29.

(5)

$$\overline{PF} = \overline{PQ}$$

$$\sqrt{(x+1)^2+y^2} = \sqrt{(x-1)^2+0^2}$$

$$x^2+2x+1+y^2 = x^2-2x+1$$

$$y^2 = -4x$$

$$r = \frac{\frac{1}{2}}{1+\cos\theta}$$

$$r = \frac{1}{2+2\cos\theta}$$

Pg 923

⑥ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ← center at origin.

$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

$$r^2\left[\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}\right] = 1$$

$$r^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}.$$

⑦

a) $r = 3\cos\theta + 4\sin\theta$

$r^2 = 3r\cos\theta + 4r\sin\theta$

$x^2 + y^2 = 3x + 4y$

$x^2 - 3x + \frac{9}{4} + y^2 - 4y + 4 = \frac{9}{4} + \frac{16}{4}$

$\left(x - \frac{3}{2}\right)^2 + (y-2)^2 = \left(\frac{5}{2}\right)^2$ circle

b) $r = \frac{1}{3\cos\theta + 4\sin\theta}$

$3r\cos\theta + 4r\sin\theta = 1$

$3x + 4y = 1$

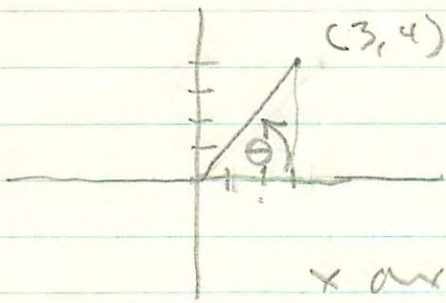
$y = -\frac{3}{4}x + \frac{1}{4}$ line.

P. 392-396

Polar Coordinates

The point (3,4) in rectangular coordinates can be located using polar coordinates.

In Polar coordinates we have a pole (a pt) and a ray (which emanates from the pole)



The pt (3,4) would be described using the length of the ray and the angle it makes with the POLAR AXIS (the x axis).

So the ray length is 5 ($\sqrt{3^2+4^2}$) and the angle can be stated as $\tan\theta = \frac{4}{3}$ so $\theta = \tan^{-1}\frac{4}{3}$ $\therefore$ $\theta \approx 53°10'$

We can see $\cos\theta = \frac{3}{5}$ $\therefore$ $3 = 5\cos\theta$ and $4 = 5\sin\theta$

we generalize to say:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

then we could describe our PT(3,4) in an infinite number of Polar representations.

$(5, 53°10' + 360°k)$ where $k \in \pm J$.

any equations in rectangular coord can be graphed in polar coord however polar coordinates are generally best suited for circular type equations.

eg ↓

Pg 392-396

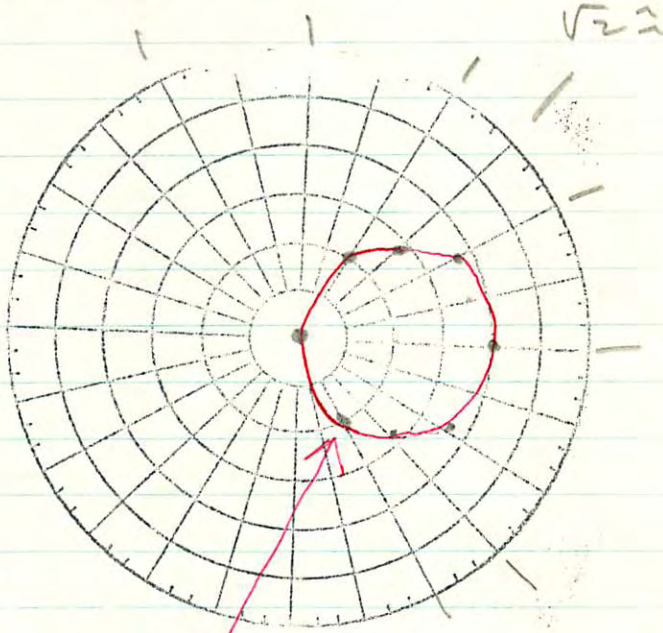
Pg 1

graph $r = 2\cos\theta$ until we have learned to recognize the general form we will need to use a table.

$\sqrt{3} \approx 1.7$

$\sqrt{2} \approx 1.4$

θ	$\cos\theta$	$2\cos\theta$
0	1	2
$\pi/6$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\pi/4$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
$\pi/3$	$\frac{1}{2}$	1
$\pi/2$	0	0
$2\pi/3$	$-\frac{1}{2}$	-1
$3\pi/4$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$
$5\pi/6$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
π	-1	-2



Notice Neg on graphing. $\left(-1, \frac{2\pi}{3}\right)$

what should this look like in rectangular coordinates? a circle, radius 1 with center at $(1,0)$ so $(x-1)^2 + y^2 = 1^2 \Rightarrow$

Don't show Book method. Let this be a reward to the student for reading. (it is a bit simplier)

$x^2 - 2x + y^2 = 0$. So we should be able to arrive at this equation strictly by substitution into $r = 2\cos\theta$.

$r = \sqrt{x^2+y^2}$ $r\cos\theta = x \Rightarrow \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$

so $\sqrt{x^2+y^2} = \frac{2x}{\sqrt{x^2+y^2}} \Rightarrow x^2+y^2 = 2x \Rightarrow$

$x^2 - 2x + y^2 = 0.$

for example I we see the graph is symetrical across the Polar axis. this is true whenever a substitution of $-\theta$ for θ does not change the equation.

Eg 2 graph $Y = 2$, $X = 3$ on same polar axis.

$r \sin\theta = 2$

$r = \frac{2}{\sin\theta}$

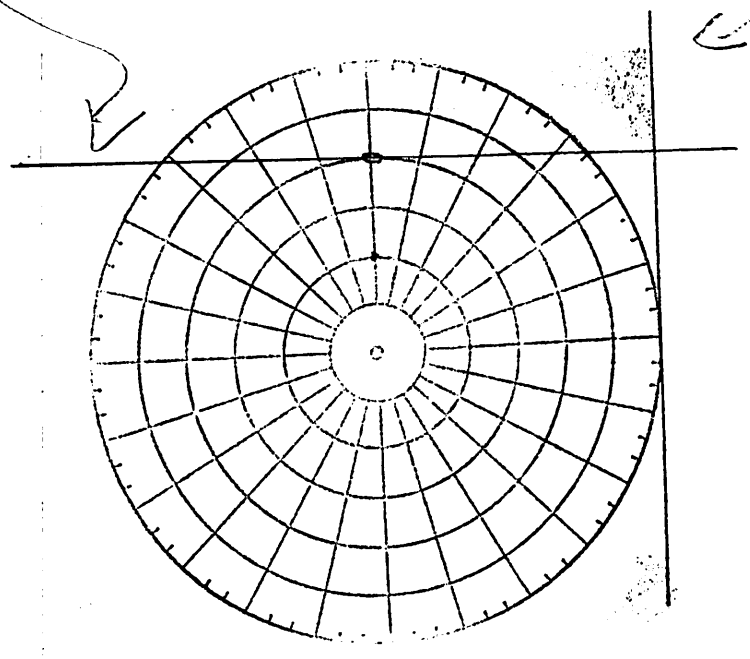
use a calculator to get values. Note $\theta \neq 0 + K\pi$

what happens as $\theta \to 0$?

$r \cos\theta = 3$

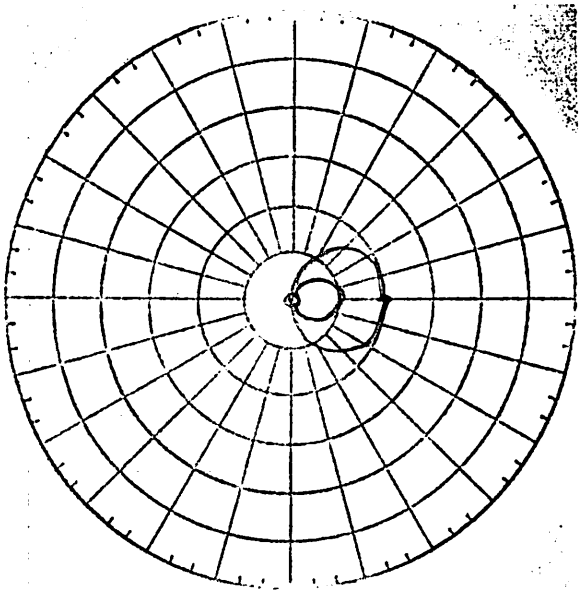
$r = \frac{3}{\cos\theta}$

here $\theta \neq \frac{\pi}{2} K$



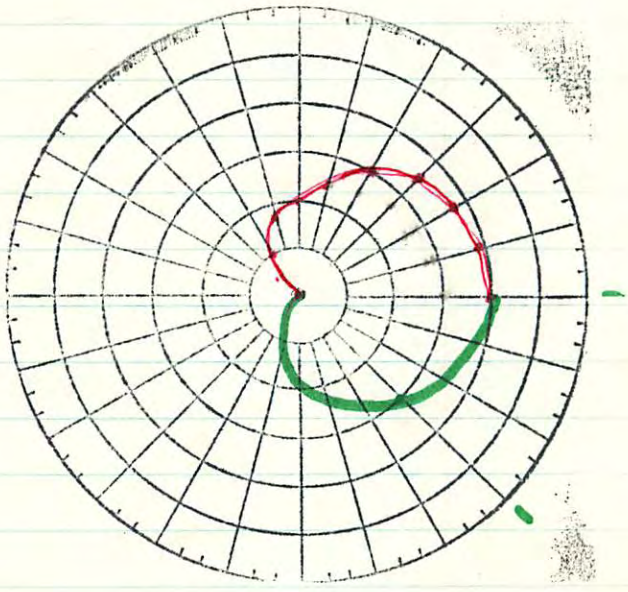
Eg 3 $r = \cos\theta$ $\quad p = \frac{1}{2}\cos\theta$ Compare to Eg 1

Circles w smaller radii



eg 4 $r = 1 + \cos\theta$ using a table (and calculator) (Symmetric about polar axis)

θ	$\cos\theta$	$1+\cos\theta$
0	1	2
15	.96	1.96
30	.87	1.87
45	.71	1.71
60	.50	1.50
75	.26	1.26
90	0	1.00
105	-.26	.74
120	-.5	.5
135	-.71	.29
150	-.87	.13
180	-1.0	0.



eg 5 $r = -2\sin^2\frac{\theta}{2}$

Before leaping in, think!

θ	θ
15	-.03
30	-.13
45	-.29
60	-.50
75	-.74
90	-1.0
105	-1.26
120	-1.50
135	-1.71
150	-1.87
175	-1.96
180	-2

Is there a trig substitution you can use?

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1}{2}(1-\cos x)}$$

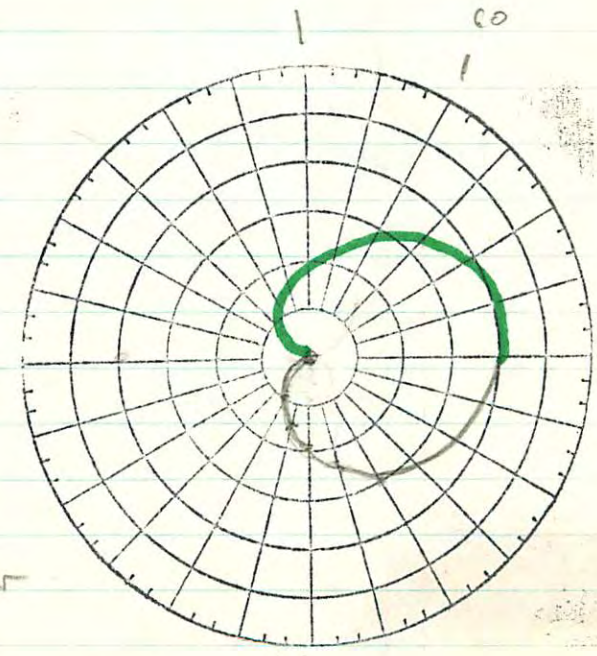
$$\sin^2\frac{x}{2} = \frac{1}{2}(1-\cos x)$$

$$2\sin^2\frac{x}{2} = 1-\cos x$$

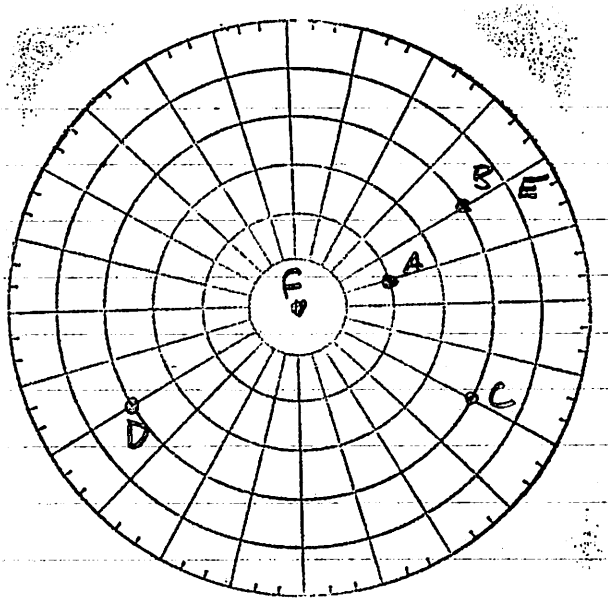
$$-2\sin^2\frac{x}{2} = -1+\cos x$$

This appears to be some type of cardioid

Reproducing 3rd column from above



It turns out to be the same as eg 4!!



① a $(1, \frac{\pi}{6})$

b) $(2, \frac{\pi}{3})$

c) $(2, -\frac{\pi}{3})$

d) $(-2, \frac{\pi}{3})$

e) $(2, \frac{7\pi}{3})$

f) $(0, \frac{\pi}{4})$

② a) $\theta = 30^\circ$

$(1 \cdot \cos 30, \sin 30^\circ) \Rightarrow (\frac{\sqrt{3}}{2}, \frac{1}{2})$

b) $(2, \frac{\pi}{3})$ $(2\cos 60, 2\sin 60) \Rightarrow (1, \sqrt{3})$

c) $(2, -\frac{\pi}{3})$ $(1, -\sqrt{3})$

D) $(-2, \frac{\pi}{3})$ $(-1, -\sqrt{3})$

E) $(2, \frac{2\pi}{3})$ $(1, \sqrt{3})$

F) $(0, \frac{\pi}{4})$ $(0, 0)$

④ a) $(\sqrt{2}, \sqrt{2})$ $(2, \frac{\pi}{4})$ $r = x^2 + y^2$ $\tan\theta = \frac{y}{x}$

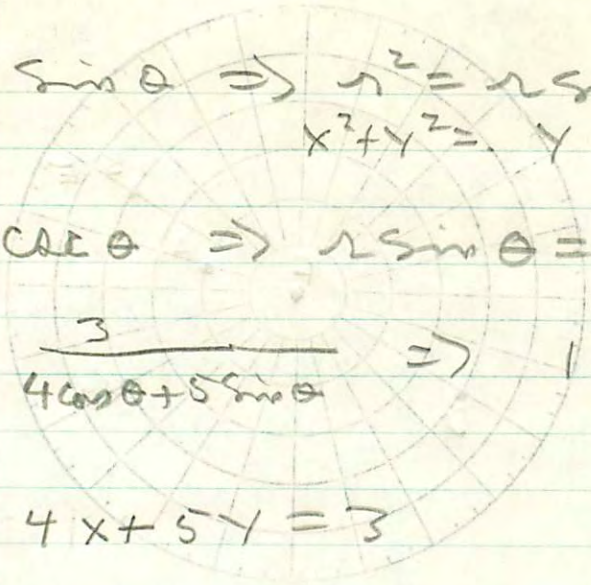
b) $(-1, \sqrt{3})$ $(2, 120^\circ)$ $(2, \frac{2\pi}{3})$

c) $(-5, 0)$ $(5, 0^\circ)$ $(5, 0)$

D) $(-\sqrt{2}, -\sqrt{2})$ $(2, 225^\circ)$ $(2, \frac{5\pi}{4})$

E) $(0, -3)$ $(3, 270^\circ)$ $(3, \frac{3\pi}{2})$

f) $(1, 1)$ $(\sqrt{2}, 45^\circ)$ $(\sqrt{2}, \frac{\pi}{4})$



(5) $r = 5\sin\theta \Rightarrow r^2 = 5r\sin\theta$

$x^2+y^2 = 5y \qquad x^2+y^2-5y=0$

(6) $r = \csc\theta \Rightarrow r\sin\theta = 1 \Rightarrow y = 1$

(7) $r = \dfrac{3}{4\cos\theta+5\sin\theta} \Rightarrow 1 = \dfrac{3}{4r\cos\theta+5r\sin\theta} \Rightarrow$

$4x+5y=3$

(8) $r = 4\cos\theta + 5\sin\theta$

$x^2+y^2 = 4x+5y$

(9) $r = \sin 2\theta$

$r^2 = 2r\sin\theta\cos\theta$

$x^2+y^2 = 2y\cos\theta \Rightarrow x^2+y^2 = \dfrac{2xy}{\sqrt{x^2+y^2}}$

$(x^2+y^2)^{3/2} = 2xy$

(10) $y = x^2$

$r\sin\theta = r^2\cos^2\theta \qquad r = \tan\theta\sec\theta$

(11) $x+2y=3 \qquad r\cos\theta + 2r\sin\theta = 3$

$r(\cos\theta+2\sin\theta) = 3$

$r = \dfrac{3}{\cos\theta+2\sin\theta}$

(12) $x^2+y^2 = 5$

$r^2 = 5$

(13) $xy = 1 \qquad r^2\sin\theta\cos\theta = 1$

$r^2 = \csc\theta\sec\theta$

$x = r\cos\theta$

(14) $x^2 + y^2 = 4x$

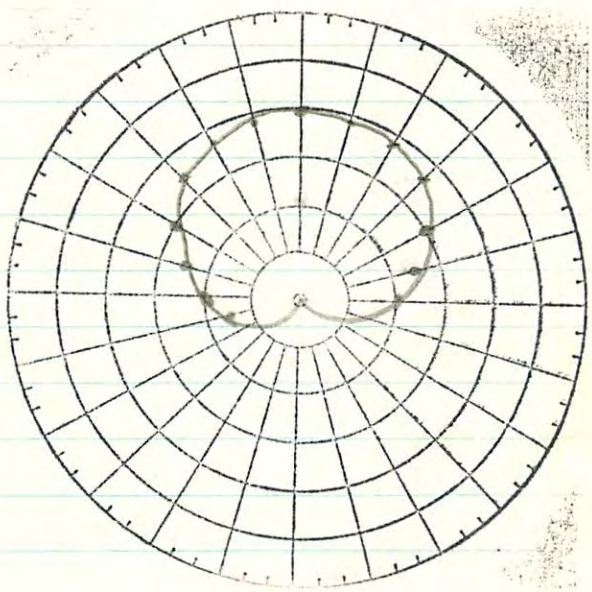
$r^2 = 4r\cos\theta \Rightarrow r = 4\cos\theta$

(15) $x = -2$

$r\cos\theta = -2 \Rightarrow r = -2\sec\theta$

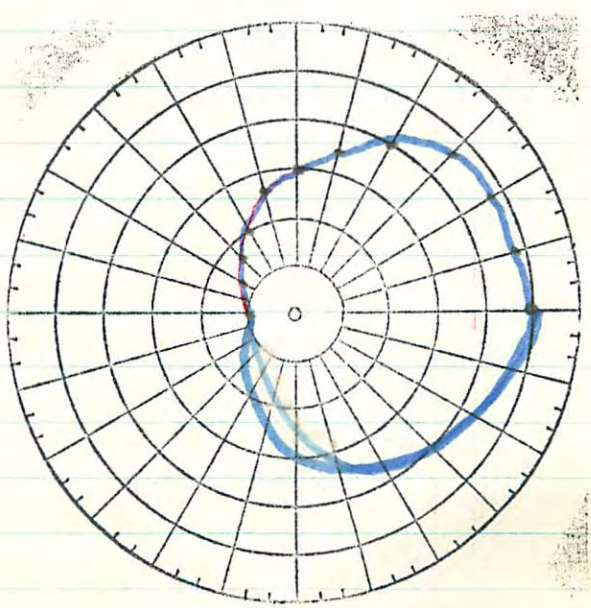
(16) $r = 1 + \sin\theta$

	θ	$\sin\theta$	$1+\sin\theta$
180	0	0	1
175	15	.26	1.26
150	30	.5	1.5
135	45	.71	1.71
120	60	.87	1.87
105	75	.97	1.97
	90	1.0	2.0
	105	.97	1.97
	195	-.26	.74



(17) $3 + 2\cos\theta = r$

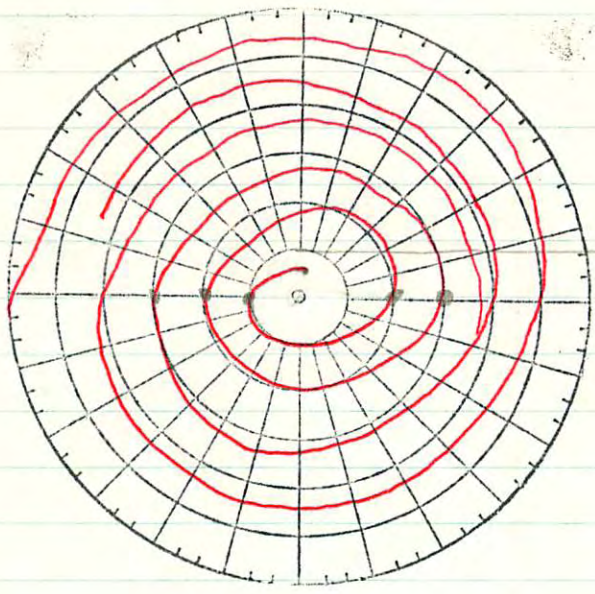
θ	$\cos\theta$	$2\cos\theta$	$3+2\cos\theta$
0	1	2	5
15	.97	1.93	4.93
30	.87	1.73	4.73
45	.71	1.41	4.41
60	.5	1.00	4.00
75	.26	.52	3.52
90	0	0	3
105	-.26	-.52	2.48
120	-.5	-1.00	2
135	-.71	-1.41	1.59
150	-.87	-1.73	1.27
175	-.97	-1.93	1.07
180	-1	-2	1.00



Too hard to see. Spirals into pole

(18) $r = \frac{1}{\theta}$ $\theta \geq 0$ change to read $r = \frac{\theta}{180}$

θ	$\frac{1}{\theta}$
15	.07
30	.03
45	.02
60	.017
75	.013
90	.011
1	1
$\frac{1}{2}$	2
$\frac{1}{3}$	3
$\frac{1}{4}$	4



(19) $r = e^{\theta}$ $\theta \geq 0$ ANother one. Too hard to see.

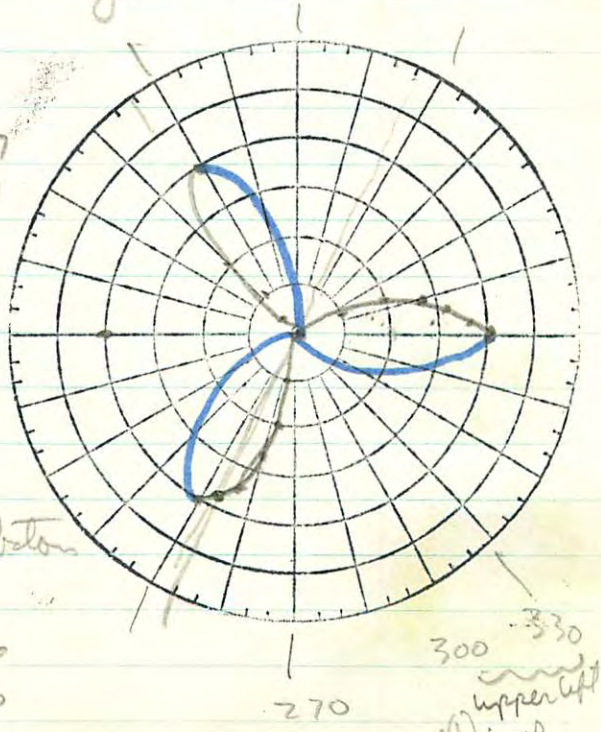
θ	e^{θ}
0	1
5	148
15	3269017

as $\theta \to 0^+$ the #s are a bit more reasonable however we still cannot plot it with any accuracy.

(20) $r = \cos 3\theta$

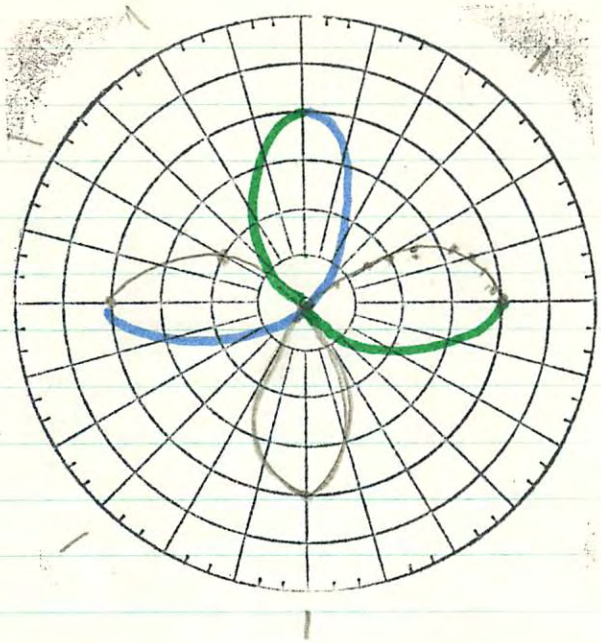
θ	3θ	$\cos 3\theta$
0	0	1
15	45	.71
30	90	0
5	15	.97
10	30	.86
20	60	.5
25	75	.26
35	105	-.26
↓		
60	Give Bottom ½	

60 180 -1
65 195 -.97
70 210 -.87
75 225 -.71
80 240 -.5
85 255 -.26
90 270 0
↓
120 upper ½ bottom
185 555
325 975 -.26
320 960 -.50
330 990



(21) $r = \cos 2\theta$

θ	2θ	$\cos 2\theta$
0	0	1
5	10	.98
10	20	.94
15	30	.87
20	40	.77
25	50	.64
30	60	.5
35	70	.34
40	80	.17
45	90	0.0
50	100	-.17
↓ 90	3rd quad ½ leaf.	
95	190	
↓ 120	4th quad ½ leaf	
125	250	Bottom ½ leaf (green)
150	300	.5
155	310	.64
180	Top ½ 2nd quad.	



22 $r = 1 + \cos\theta$ $r = 2\cos\theta$

Intersect at $(0,0)$ $(0,2)$ (cartesian)
Pole $(2,0)$ (polar)

$1 + \cos\theta \overset{\text{set}}{=} 2\cos\theta$

$\cos\theta = 1$

$\theta = 0$

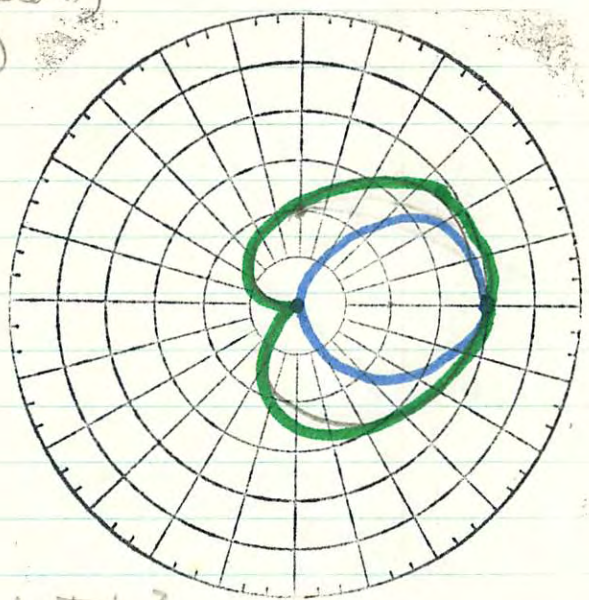
Time enough. Both intersections are NOT obtained in this manner.

$r = 1 + \cos\theta$

$2r = 2 + 2\cos\theta$

$2r = 2 + r$

$r = 2$ is this the other solution?



23 $r = 2\sin\theta$ $r = \cos 2\theta$

pole, $(1, 90^\circ)$ $\left(\frac{1}{2}, 30^\circ\right)$ $\left(\frac{1}{2}, 150^\circ\right)$

$\sin\theta \overset{\text{set}}{=} \cos 2\theta$

$\sin\theta = \cos^2\theta - \sin^2\theta$

$\sin\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$

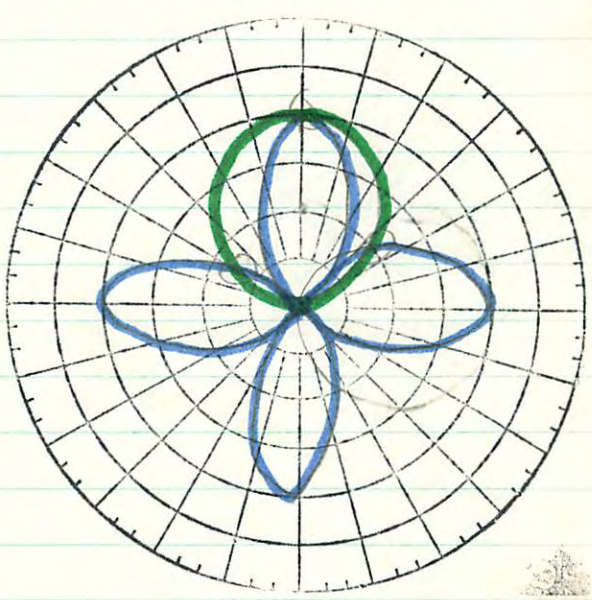
$(2\sin\theta - 1)(\sin\theta + 1)$

$\sin\theta = \frac{1}{2}$ $\sin\theta = -1$

$\theta = 30^\circ$ $\theta = 270^\circ$

$(-1, 270)$

guess for symmetry.



The method for complete solutions to polar equations is covered in advanced calculus.